On defining relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras *

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Introduction. In this paper, we give defining relations of the affine Lie superalgebras and defining relations of a super-version of the Drinfeld[D1]-Jimbo[J] affine quantized enveloping algebras. As a result, we can exactly define the affine quantized universal enveloping superalgebras with generators and relations. Moreover we give a Drinfeld's realization of $U_h(\hat{sl}(m|n)^{(1)})$.

For the Kac-Moody Lie algebra G, Gabber-Kac [GK] proved the Serre theorem which states that G is defined with the Chevalley generators H_i, E_i, F_i $(1 \leq i \leq \mathrm{rank}G)$ and relations

$$[H_i, H_j] = 0,\ [H_i, E_j] = (\alpha_i, \alpha_j)E_j,\ [H_i, F_j] = -(\alpha_i, \alpha_j)F_j,$$

$$[E_i, F_j] = \delta_{ij}H_i,$$
$$ad(E_i)^{1-a_{ij}}(E_j) = 0,\ ad(F_i)^{1-a_{ij}}(F_j) = 0$$

where $\{\alpha_i(1 \leq i \leq \mathrm{rank}G)\}$ are simple roots of G, $(\ ,\ )$ is an invariant form of G and (a_{ij}) is the Cartan matrix of G. We call these relations Serre's relations.

Kac [K2] classified the finite dimensional simple Lie superalgebras, which are $sl(m|n)$, $osp(m|n)$, $D(2,1;x)$ $(x \neq 0,1)$, F_4 and G_3. Van de Leur [VdL] classified the Kac-Moody Lie superalgebras $\mathcal{G}$ of finite growth, which are the finite dimensional simple Lie superalgebras and:

$$\hat{sl}(m|n)^{(i)}\ (i = 1,2,4),\ \widehat{osp}(m|n)^{(i)}\ (i = 1,2),$$
$$D(2,1;x)^{(1)}\ (x \neq 0,1),\ F_4^{(1)},\ G_3^{(1)}.$$

In this paper we call complex infinite dimensional Kac-Moody Lie superalgebras of finite growth affine Lie superalgebras.

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Our first result is to give a Serre theorem for the affine Lie superalgebra $\mathcal{G}$, i.e., to give defining relations between the Chevalley generators H_i, E_i, F_i. We give the defining relations associated to each Cartan matrix of $\mathcal{G}$. (In general, $\mathcal{G}$ does not have a unique Cartan matrix.) To do this, we use Weyl-group-type isomorphisms $\{L_i\}$ between $\mathcal{G}$. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. We note that the Cartan matrix defined for $\mathcal{H}$ does not necessarily coincide with the one defined for $L_i(\mathcal{H})$, though $\{L_i\}$ are introduced as counterparts of the inner automorphisms $\{\exp(-\mathrm{ad}F_i)\exp(-\frac{2}{(\alpha_i,\alpha_i)}\mathrm{ad}E_i)\exp(-\mathrm{ad}F_i)\ (1 \leq i \leq \mathrm{rank}G)\}$ of the Kac-Moody Lie algebra G. We introduce another Lie superalgebra $\bar{\mathcal{G}}$ associated to each $\mathcal{G}$. We define the Lie superalgebras $\bar{\mathcal{G}}$ by a universal condition that $\{L_i\}$ can be lifted to isomorphisms $\{\bar{L}_i\}$ between $\bar{\mathcal{G}}$. We directly calculate defining relations of $\bar{\mathcal{G}}$. In the case of the Kac-Moody Lie algebra G, defining relations of $\bar{G}$ are given by Serre's relations. However, for $\mathcal{G}$, we need other relations such as

$$[[E_i, E_j], [E_j, E_k]] = 0 \quad \text{for}\, (\alpha_i, \alpha_k) = (\alpha_j, \alpha_j) = 0,\, (\alpha_i, \alpha_j) = -(\alpha_j, \alpha_k) \neq 0.$$

There is an epimorphism $j : \bar{\mathcal{G}} \to \mathcal{G}$ satisfying $j \circ L_i = \bar{L}_i \circ j$. In the case of the Kac-Moody Lie algebra, most of the proof of the Gabber-Kac theorem [GK] was to prove $\ker j = 0$.

However, in the case of the affine Lie superalgebras $\mathcal{G}$, $\bar{\mathcal{G}} \neq \mathcal{G}$ if and only if $\mathcal{G} = \hat{sl}(m|n)^{(i)}$ $(i = 1, 2, 4)$ and $m = n$. (In the case of $\mathcal{G} = \hat{sl}(m|m)^{(1)}$, $\bar{\mathcal{G}} = sl(m|m) \otimes C[t, t^{-1}] \oplus Cc \oplus Cd$ and $\ker j = I \otimes C[t, t^{-1}]$ where I is the unit matrix.) Nevertheless, we can also look for defining relations of $\mathcal{G} = \hat{sl}(m|m)^{(i)}$ because we have concretely known $\hat{sl}(m|m)^{(i)}$.

Our second result is to give relations of quantized universal enveloping superalgebras $U_h(\mathcal{G})$ such that, after $h \to 0$, the relations become the defining relations of $\bar{\mathcal{G}}$ obtained as our first result. Here $U_h(\mathcal{G})$ is an h-adic topological $C[[h]]$-Hopf superalgebra introduced in [Y1]. In [Y1], we showed an existence of a non-degenerate symmetric bilinear form defined on a Borel part of $U_h(\mathcal{G})$.

Applying the Drinfeld's quantum double construction to $U_h(\mathcal{G})$ by using the bilinear form, we can see that $U_h(\mathcal{G})$ is topologically free and that the universal R-matrix of $U_h(\mathcal{G})$ exists.

Since $U_0(\mathcal{G}) = U_h(\mathcal{G})/hU_h(\mathcal{G})$ is a cocommutative Hopf C-superalgebra, applying Milnor-Moor theorem [MM] to $U_0(\mathcal{G})$, we see that $U_0(\mathcal{G})$ is a universal enveloping superalgebra $U(\mathcal{G}_0)$ of the Lie superalgebra $\mathcal{G}_0 = \mathcal{P}(U_0(\mathcal{G}))$ of primitive elements of $U_0(\mathcal{G})$. By definition of $\mathcal{G}$ as the Kac-Moody Lie superalgebra, $\mathcal{G}$ must be a quotient of $\mathcal{G}_0$. On the other hand, we see that $\mathcal{G}_0$ is a quotient of $\bar{\mathcal{G}}$ by our second result. Hence, if $\mathcal{G} = \bar{\mathcal{G}}$, we see that $U_0(\mathcal{G})$ coincides with $U(\mathcal{G})$ and that our relations must be defining relations of $U_h(\mathcal{G})$ by the topologically freedom of $U_h(\mathcal{G})$.

Finally, we calculate relations of $U_h(\hat{sl}(m|m)^{(1)})$ which become ones generating $\ker j$ after $h \to 0$, while showing Drinfeld's realization of $U_h(\hat{sl}(m|n)^{(1)})$ for general m, n. Gathering up the relations and the ones obtained as our

second result, we get defining relations of $U_h(\hat{sl}(m|m)^{(1)})$. To do these, we introduce a Braid group action on $U_h(\hat{sl}(m|n)^{(1)})$ which become the action on $\hat{sl}(m|n)^{(1)}$ defined by $\{L_i\}$ after $h \to 0$, and follow Beck's argument [B]. We won't consider $U_h(\hat{sl}(m|m)^{(2)})$ or $U_h(\hat{sl}(m|m)^{(4)})$.

Results in this paper have already been announced in [Y2]. The same results for the finite dimensional $A - G$ type simple Lie superalgebras have already been given in [Y1].

1 Preliminary

1.1. In §1, we mainly refer to [K1-2] and [VdL].

Let $\mathcal{G}$ be a C-vector space with a direct sum decomposition $\mathcal{G} = \mathcal{G}(0) \oplus \mathcal{G}(1)$. For $X \in \mathcal{G}$, $p(X) \in \{0, 1\}$ means that $X \in \mathcal{G}(p(X))$. We call $p(X)$ the parity of X. A Lie superalgebra $\mathcal{G}$ is defined with the bilinear map $[\ ,\ ] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

$$[X, Y] = -(-1)^{p(X)p(Y)}[Y, X],$$

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{p(X)p(Y)}[Y, [X, Z]].$$

If a bilinear form $(\ |\ ) : \mathcal{G} \times \mathcal{G} \to C$ satisfies $(X|Y) = (-1)^{p(X)p(Y)}(Y|X)$ and $([X, Y]|Z) = (X|[Y, Z])$, then we call it an invariant form.

A Lie superalgebra $\hat{\mathcal{G}} = \mathcal{G} \otimes_C C[t, t^{-1}] \oplus Cc \oplus Cd$ is defined by

$$[X \otimes t^m + a_1c + b_1d,\ Y \otimes t^n + a_2c + b_2d]$$

$$= [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}(X|Y)c + b_1nY \otimes t^n - b_2mX \otimes t^m.$$

where $\hat{\mathcal{G}}(0) = \mathcal{G}(0) \otimes_C C[t, t^{-1}] \oplus Cc \oplus Cd$ and $\hat{\mathcal{G}}(1) = \mathcal{G}(1) \otimes_C C[t, t^{-1}]$.

Let $\gamma : \mathcal{G} \to \mathcal{G}$ be an automorphism of finite order r (i.e. $\gamma([X, Y]) = [\gamma(X), \gamma(Y)]$). Put

$$\mathcal{G}^\gamma_n = \{X \in \mathcal{G} | \gamma(X) = (exp\frac{2\pi\sqrt{-1}}{n})X\} \quad (0 \leq n < r).$$

Then $\mathcal{G}^\gamma_0$ is a subalgebra of $\mathcal{G}$ and $\mathcal{G}^\gamma_i$ $(1 \leq i \leq r-1)$ is the $\mathcal{G}^\gamma_0$-module. We can define a subalgebra $\hat{\mathcal{G}}^{(\gamma)}$ by

$$\hat{\mathcal{G}}^{(\gamma)} = \bigoplus_{n=0}^{r-1}(\bigoplus_{m \in Z} \mathcal{G}^\gamma_n \otimes t^{mr+n}) \oplus Cc \oplus Cd\,.$$

Obviously $\hat{\mathcal{G}}^{(1)} = \hat{\mathcal{G}}$.

1.2. Here we introduce a definition of the Kac-Moody Lie superalgebra in an abstract manner similar to the abstract definition of the Kac-Moody Lie algebra given in [K1;1.3]. Let $\mathcal{E}$ be a finite dimensional C vector space with a

nondegenerate symmetric bilinear form $(\,,\,) : \mathcal{E}\times\mathcal{E} \to C$. Let $\Pi = \{\alpha_0, ..., \alpha_1\}$ be a finite linearly independent subset of $\mathcal{E}$. We call an element $\alpha_i \in \Pi$ the simple root. Let $p : \Pi \to Z/2Z$ be a function. We call p the parity function. Put $\mathcal{H} = \mathcal{E}^*$. We call $\mathcal{H}$ the Cartan subalgebra. We identify an element $\gamma \in \mathcal{E}$ with $H_\gamma \in \mathcal{H}$ satisfying $\delta(H_\gamma) = (\delta, \gamma)$ $(\delta \in \mathcal{E})$. For a datum $(\mathcal{E}, \Pi, p)$, we define a Lie superalgebra $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(\mathcal{E}, \Pi, p)$ with the generators E_i, F_i $(0 \le i \le n)$, $H \in \mathcal{H}$, the parities $p(E_i) = p(F_i) = p(\alpha_i)$, $p(H) = 0$ and the relations:

$$[H, H'] = 0 \quad (H, H' \in \mathcal{H}), \tag{1.2.1}$$

$$[H, E_i] = \alpha_i(H)E_i,\ [H, F_i] = -\alpha_i(H)F_i, \tag{1.2.2}$$

$$[E_i, F_j] = \delta_{ij}H_{\alpha_i}. \tag{1.2.3}$$

Then we have the triangular decomposition $\widetilde{\mathcal{G}} = \widetilde{\mathcal{N}}^+ \oplus \mathcal{H} \oplus \widetilde{\mathcal{N}}^-$. Here $\widetilde{\mathcal{N}}^+$ (resp. $\widetilde{\mathcal{N}}^-$) is the free superalgebra with generators E_i (resp. F_i). Define the quotient Lie superalgebra $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$ of $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(\mathcal{E}, \Pi, p)$ by

$$\mathcal{G}(\mathcal{E}, \Pi, p) = \widetilde{\mathcal{G}}(\mathcal{E}, \Pi, p)/r\ .$$

where $r = r(\mathcal{E}, \Pi, p)$ is the ideal which is maximal of the ideals r_1 satisfying $r_1 \cap \mathcal{H} = 0$. We call $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$ the Kac-Moody Lie superalgebra. We have the triangular decomposition

$$\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$$

where $\mathcal{N}^+$ and $\mathcal{N}^-$ are the subalgebras of $\mathcal{G}$ generated by E_i and F_i respectively. Let $r_\pm = r \cap \widetilde{\mathcal{N}}^\pm$. Then we have $r = r_- \oplus r_+$ and $\mathcal{N}^\pm = \widetilde{\mathcal{N}}^\pm/r_\pm$. We also have $\widetilde{\mathcal{N}}^+ \cong \widetilde{\mathcal{N}}^-$ $(E_i \leftrightarrow F_i)$. Let $\mathcal{G} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\mathcal{E}} \mathcal{G}_\alpha)$ be the root space decomposition where $\mathcal{G}_\alpha = \{X \in \mathcal{G} | [H, X] = \alpha(H)X\, (H \in \mathcal{H})\}$. Let $r_\alpha = r \cap \mathcal{G}_\alpha$. We put $\Phi = \Phi(\mathcal{E}, \Pi, p) = \{\alpha \in \mathcal{E} | \mathcal{G}_\alpha \ne 0\}$. Let $P_+ = Z_+\alpha_0 \oplus \cdots \oplus Z_+\alpha_n$ and $\Phi_+ = \Phi \cap P_+$, $\Phi_- = -\Phi_+$. Then we have $\Phi = \Phi_+ \cup \Phi_-$. Clearly, we have $r = \bigoplus_{\alpha\in\Phi} r_\alpha$.

For $\beta, \alpha \in P_+$, we say $\beta \le \alpha$ if $\alpha - \beta \in P_+$. Let $r_{+,\le\alpha}$ be the ideal of $\widetilde{\mathcal{N}}^+$ generated by r_β with $\beta \le \alpha$. Then $r_+ = \cup_{\gamma\in P_+} r_{+,\le\gamma}$. By the same argument in [K1], we have:

Proposition 1.2.1. *For $(\mathcal{E}, \Pi, p)$, let $\rho \in \mathcal{E}$ be an element such that $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$. If $\alpha \in P_+$ satisfies $(\alpha, \alpha) \ne 2(\rho, \alpha)$, then r_α is included in the ideal of $\widetilde{\mathcal{N}}^+$ generated by r_β such that $\beta \ge \alpha$. In particular,*

$$r_+ = \bigcup_{\gamma\in P_+,\, (\gamma,\gamma)=2(\rho,\gamma)} r_{+,\le\gamma} \quad . \tag{1.2.4}$$

Lemma 1.2.1. *For $\alpha_i \in \Pi$, we have $\dim \mathcal{G}_{\alpha_i} = 1$. If $p(\alpha_i) = 1$ and*

$(\alpha_i, \alpha_i) \neq 0$, *then* $\dim \mathcal{G}_{2\alpha_i} = 1$.
Proof. By $[E_i.F_i] = H_{\alpha_i} \neq 0$, $\dim \mathcal{G}_{\alpha_i} \neq 0$. Hence $\dim \mathcal{G}_{\alpha_i} = 1$. Similarly we can get a proof of the latter half.

Q.E.D.

1.3. Here we introduce the Dynkin diagram Γ for a datum $(\mathcal{E}, \Pi, p)$. We first prepare the three-type dots:

$$\bigcirc\ , \qquad \otimes\ , \qquad \bullet\ .$$

We call those the white dot, the gray dot and the black dot respectively. To the i-th simple root α_i, we put the corresponding i-th dot defined such that:

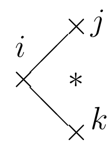
$$\begin{array}{ll} \bigcirc & \text{if } (\alpha_i, \alpha_i) \neq 0 \text{ and } p(\alpha_i) = 0\,, \\ \otimes & \text{if } (\alpha_i, \alpha_i) = 0 \text{ and } p(\alpha_i) = 1\,, \\ \bullet & \text{if } (\alpha_i, \alpha_i) \neq 0 \text{ and } p(\alpha_i) = 1\,. \end{array}$$

The dot $\times$ stands for $\bigcirc$ or $\otimes$. The dot $\odot$ stands for $\bigcirc$ or $\bullet$. We write a $|(\alpha_i, \alpha_j)|$-line between the i-th dot and the j-th dot or write as follows:

$$\underset{\cdot}{i} \ \overset{(\alpha_i, \alpha_j)}{\overline{\qquad\qquad\qquad}} \ \underset{\cdot}{j}$$

Moreover we add a pile pointing to the smaller of $|(\alpha_i, \alpha_i)|$ and $|(\alpha_j, \alpha_j)|$. If $|(\alpha_i, \alpha_i)| = 0$ or $|(\alpha_j, \alpha_j)| = 0$, then we sometimes omit the pile. The

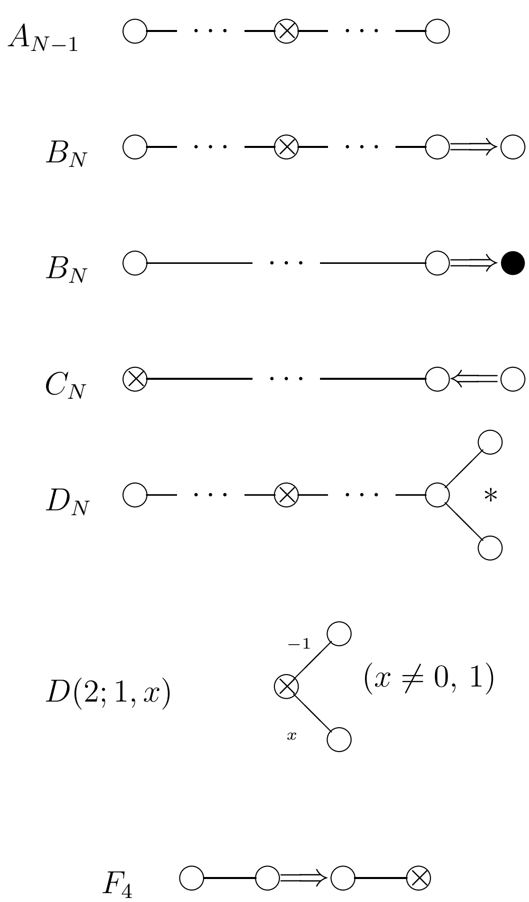
semilines $\begin{array}{c}\times\\ \vdots\\ \times\end{array}$ stands for $\begin{array}{c}\bigcirc\\ \\ \bigcirc\end{array}$ or $\begin{array}{c}\otimes\\ \|\\ \otimes\end{array}$. If $(\alpha_i, \alpha_j) \neq 0$, $(\alpha_i, \alpha_k) \neq 0$ and

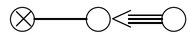
$\dfrac{|(\alpha_i,\alpha_j)|}{(\alpha_i,\alpha_j)}\cdot\dfrac{|(\alpha_i,\alpha_k)|}{(\alpha_i,\alpha_k)}=1$, then we put $*$ in a sector enclosed by an edge between i-th and j-th dots and an edge between i-th and k-th dots. Namely we describe the situation as 

. However we sometimes omit $*$.

1.4. We have already known:
Theorem 1.4.1[K2]. *The Kac-Moody Lie superalgebra $\mathcal{G}(\mathcal{E},\Pi,p)$ is finite dimensional as a C-vector space if and only if the datum is one of the following Dynkin diagram. (In any diagram, there is an only one dot whose parity is odd.)*

Diagram 1.4.1



G_3 

1.5. In 1.5-13, we give a concrete form of finite dimensional $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ of $(\mathcal{E}_0, \Pi_0, p_0)$ of $ABCD$-type, which is given by using matrices.

Let $\tilde{\mathcal{E}}_0$ is an $\tilde{N}$-dimensional C-vector space with a nondegenerate symmetric bilinear form $(\ ,\ ) : \tilde{\mathcal{E}}_0 \times \tilde{\mathcal{E}}_0 \to C$. Let $e \in \{\pm 1\}$. Let $\bar{\varepsilon}_i$ $(1 \le n \le N)$ be the orthogonal basis of $\tilde{\mathcal{E}}_0$ satisfying

$$(\bar{\varepsilon}_i, \bar{\varepsilon}_j) = e \cdot \delta_{ij} \cdot (-1)^{\tilde{p}(i)} \ .$$

where $\tilde{p}(i)$ is 0 or 1. Let $\bar{d}_i = (\bar{\varepsilon}_i, \bar{\varepsilon}_i)$. Let $gl(\tilde{\mathcal{E}}_0, e) = gl(\tilde{\mathcal{E}}_0)$ be the C-linear space of $\tilde{N} \times \tilde{N}$ -matrices. Put $E_{ij} = (\delta_{xi}, \delta_{yi})_{1 \le x, y \le \tilde{N}} \in gl(\tilde{\mathcal{E}}_0)$ $(1 \le i,\, j \le \tilde{N})$. We regard $gl(\tilde{\mathcal{E}}_0, e)$ as a superspace with a parity p defined by $p(E_{ij}) = (-1)^{(\tilde{p}(i)+\tilde{p}(j))}$. Then $gl(\tilde{\mathcal{E}}_0, e)$ can be regarded as a Lie superalgebra defined by

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX \, .$$

Define a C-linear map $str : gl(\mathcal{E}_0, e) \to C$ by $str(E_{ij}) = \delta_{ij} \cdot \bar{d}_i$. Let $sl(\mathcal{E}_0, e)$ denote the subalgebra of $gl(\mathcal{E}_0, e)$ of the elements $X \in gl(\mathcal{E}_0, e)$ satisfying $str(X) = 0$. Let $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ be a finite dimensional Kac-Moody Lie superalgebra. Let $\gamma : \mathcal{G}(\mathcal{E}_0, \Pi_0, p_0) \to \mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ be an automorphism of finite order r. In 1.6-13, We give a concrete form of an affine $ABCD$-type superalgebra $\mathcal{G}(\mathcal{E}, \Pi, p)$ arising from $\hat{\mathcal{G}}(\mathcal{E}_0, \Pi_0, p_0)^{(\gamma)}$. Let $(\mathcal{E}_0^\gamma, \Pi_0^\gamma, p_0^\gamma)$ be the datum of $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)_0^\gamma$, i.e., $\mathcal{G}(\mathcal{E}_0^\gamma, \Pi_0^\gamma, p_0^\gamma) = \mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)_0^\gamma$. For γ, we define the datum $(\mathcal{E}, \Pi, p) = (\mathcal{E}_0, \Pi_0, p_0)^{(\gamma)}$ of affine type as follows:

(i) $\qquad \mathcal{E} = \mathcal{E}_0^\gamma \oplus C\delta \oplus C\Lambda_0$

where $(x, \delta) = (x, \Lambda_0) = 0$ $(x \in \mathcal{E}_0^\gamma)$, $(\delta, \delta) = (\Lambda_0, \Lambda_0) = 0$ and $(\delta, \Lambda_0) = 1$.

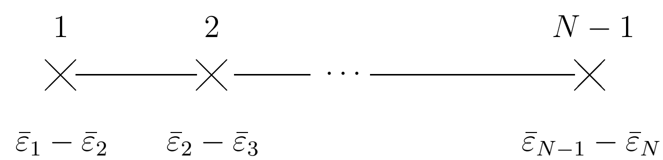
(ii) For the lowest weight ψ of $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)_1^\gamma$, let $\alpha_0 = \delta + \psi$ and $\Pi = \{\alpha_0\} \cup \Pi_0^\gamma$.

(iii) Let $e_\psi \in \mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)_1^\gamma$ be a weight vector of ψ. We define the parity $p(\alpha_0)$ of α_0 by the parity of e_ψ in $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$. We also define $p(\alpha_i) = p_0^\gamma(\alpha_i)$ $(\alpha_i \in \Pi_0^\gamma)$

1.6. Here we put $N = \tilde{N}, \mathcal{E}_0 = \tilde{\mathcal{E}}_0,\ n = N - 1,\ N \ge 2$ and $e = 1$. Let $(\mathcal{E}_0, \Pi_0, p_0)$ be the datum whose Dynkin diagram is:

Diagram 1.6.1

A



Then we can identify $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ with $gl(\mathcal{E}_0, e)$ where $H_{\bar{\varepsilon}_j} = \bar{d}_j E_{jj},\ E_i = E_{ii+1},\ F_i = \bar{d}_i E_{i+1i}$. We also note that $H_{\alpha_i} = \bar{d}_i E_{ii} - \bar{d}_{i+1} E_{i+1i+1}$.

We also note the lowest root of $gl(\mathcal{E}_0, e)$ is $\theta = \bar{\varepsilon}_N - \bar{\varepsilon}_1$. Lowest and highest root vectors $E_\theta,\ F_\theta\ (\in gl(\mathcal{E}_0, e))$ satisfying $[E_\theta, F_\theta] = H_\theta (= \bar{d}_N E_{NN} - \bar{d}_1 E_{11})$ are given by

$$F_\theta = \bar{d}_N E_{1N}, E_\theta = E_{N1}.$$

Proposition 1.6.1. If $\sum_{i=1}^N \bar{d}_i \neq 0$, then $sl(\mathcal{E}_0, e)$ is the simple Lie superalgebra. If $\sum_{i=1}^N \bar{d}_i = 0$, then the quotient $sl(\mathcal{E}_0, e)/\mathcal{I}$ is the simple Lie superalgebra where

$$\mathcal{I} = C \cdot \textstyle\sum_{i=1}^{N-1}((\sum_{j=1}^i \bar{d}_j) \cdot H_{\alpha_i}) = C \cdot \sum_{i=1}^N E_{ii} .$$

Proposition 1.6.2. *Assume $N \geq 3$ if $p(\alpha_0) = 1$. Let $(\mathcal{E}_0, \Pi_0, p_0)$ be the datum of Diagram 1.6.1 and $(\mathcal{E}, \Pi, p) = (\mathcal{E}_0, \Pi_0, p_0)^{(I)}$. Let $\bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p) = \hat{sl}(\mathcal{E}_0, e) + \mathcal{E}_0\ (\subset\ \hat{gl}(\mathcal{E}_0, e))$. There is an epimorphism $j : \bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p) \to \mathcal{G}(\mathcal{E}, \Pi, p)$ defined by letting $j(H \otimes 1 + ac + bd) = H + aH_\delta + bH_{\Lambda_0}\ (H \in \mathcal{H})$ and $j(E_\theta \otimes t) = E_0,\ j(F_\theta \otimes t^{-1}) = F_0$.*

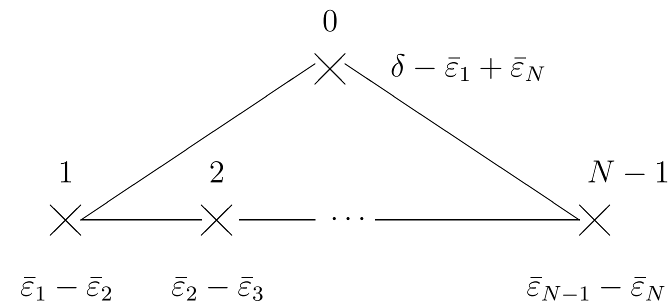
If $\sum_{i=1}^N \bar{d}_i \neq 0$, then j is an isomorphism. If $\sum_{i=1}^N \bar{d}_i = 0$, then

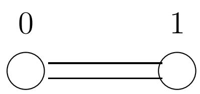
$$\ker j = \oplus_{i \neq 0} \mathcal{I} \otimes t^i.$$

Example 1.6.1. The Dynkin diagram of $(\mathcal{E}, \Pi, p)$ in Proposition 1.6.2 is:

Diagram 1.6.2. $(N \geq 2)$

(AA)





$(\sum_{i=0}^{N-1} p(\alpha_i) \equiv 0)$.

1.7. Here we assume

$$\tilde{N} = \begin{cases} 2N+1 & \text{if } \tilde{N} \text{ is odd,} \\ 2N & \text{if } \tilde{N} \text{ is even.} \end{cases}$$

We assume $\tilde{N} \geq 3$. Let $i' = \tilde{N} + 1 - i$. We also assume

$$\tilde{p}(i) = \tilde{p}(i') \ (1 \leq i \leq \tilde{N})$$

and

$$\tilde{p}(N+1) = 0 \text{ if } \tilde{N} \text{ is odd.}$$

Let g_i $(1 \leq i \leq \tilde{N})$ be such that $g_i \in \{\pm 1\}$ and $g_i g_{i'} = (-1)^{\tilde{p}(i)}$. We assume that $g_{N+1} = 1$ if $\tilde{N}$ is odd.

Let Ω be an automorphism of $gl(\tilde{\mathcal{E}}_0, e)$ of order 2 defined by:

$$\Omega(X)_{ij} = -(-1)^{(\tilde{p}(i)\tilde{p}(j)+\tilde{p}(j))} g_i g_j X_{j'i'}\,.$$

Denote $sl(\tilde{\mathcal{E}}_0, e)_0^{\Omega}$ by $osp(\tilde{\mathcal{E}}_0, e)$. We can identify $osp(\tilde{\mathcal{E}}_0, e)$ with a finite dimensional Kac-Moody Lie superalgebra $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ of a datum $(\mathcal{E}_0, \Pi_0, p_0)$.

Here we note $n = N$. In 1.8-11 we will give:

(1) the datum $(\mathcal{E}_0, \Pi_0, p_0)$.

(2) the lowest root θ of $sl(\tilde{\mathcal{E}}_0, e)_0^{\Omega}$, a lowest root vector E_θ and a highest root vector F_θ such that $[E_\theta, F_\theta] = H_\theta$.

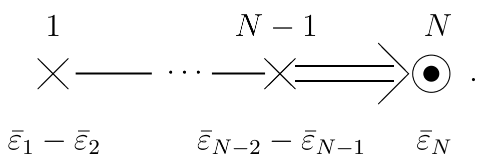
(3) the lowest weight ψ of $sl(\tilde{\mathcal{E}}_0, e)_1^{\Omega}$, a lowest weight vector E_ψ and a highest weight vector F_ψ such that $[E_\psi, F_\psi] = H_\psi$.

(4) $\mathcal{G}(\mathcal{E}, \Pi, p)$ and $\bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p)$ arising from $\widehat{osp}(\tilde{\mathcal{E}}_0, e)^{(I)}$ and $\hat{sl}(\tilde{\mathcal{E}}_0, e)^{(\Omega)}$.

1.8. B-type. If $\tilde{N}$ is odd, then the Dynkin diagram of $(\mathcal{E}_0, \Pi_0, p_0)$ is:

Diagram 1.8.1.

B



Here $H_{\bar\varepsilon_j} = \bar{d}_j(E_{jj} - E_{j'j'})$ $(1 \le j \le N)$, $E_i = E_{ii+1} - (-1)^{\tilde{p}(i)\tilde{p}(i+1)}(-1)^{\tilde{p}(i+1)} g_i g_{i+1} E_{(i+1)'i'}$ $(1 \le i \le N-1)$, $E_N = E_{NN+1} - g_N E_{(N+1)'N'}$, $F_i = e \cdot \{(-1)^{\tilde{p}(i)} E_{i+1i} - (-1)^{\tilde{p}(i)\tilde{p}(i+1)} g_i g_{i+1} E_{i'(i+1)'}\}$ $(1 \le i \le N-1)$, $F_N = e \cdot \{(-1)^{\tilde{p}(N)} E_{N+1N} - g_N E_{N'(N+1)'}\}$.

Moreover we have:

If $p(\alpha_1) + \cdots + p(\alpha_{N-1}) \equiv 0$, then

$$\theta = -\bar\varepsilon_1 - \bar\varepsilon_2, F_\theta = e \cdot \{(-1)^{\tilde{p}(1)} E_{21'} - (-1)^{\tilde{p}(1)\tilde{p}(2)} g_2 g_{1'} E_{12'}\},$$

$$E_\theta = E_{1'2} - (-1)^{\tilde{p}(1)\tilde{p}(2)}(-1)^{\tilde{p}(2)} g_{1'} g_2 E_{2'1},$$

$$\psi = -2\bar\varepsilon_1, F_\psi = e \cdot 2(-1)^{\tilde{p}(1)} E_{11'}, E_\psi = E_{1'1}.$$

If $p(\alpha_1) + \cdots + p(\alpha_{N-1}) \equiv 1$, then

$$\theta = -2\bar\varepsilon_1, F_\theta = e \cdot 2(-1)^{\tilde{p}(1)} E_{11'}, E_\theta = E_{1'1}.$$

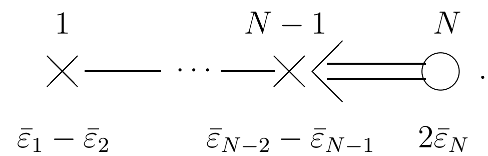
$$\psi = -\bar\varepsilon_1 - \bar\varepsilon_2, F_\psi = e \cdot \{(-1)^{\tilde{p}(1)} E_{21'} + (-1)^{\tilde{p}(1)\tilde{p}(2)} g_2 g_{1'} E_{12'}\},$$

$$E_\psi = E_{1'2} + (-1)^{\tilde{p}(1)\tilde{p}(2)}(-1)^{\tilde{p}(2)} g_{1'} g_2 E_{2'1},$$

1.9. C-type. If $\tilde{N}$ is even and $\tilde{p}(N) = 1$, $e = -\bar{d}_N$, then the Dynkin diagram of $(\mathcal{E}_0, \Pi_0, p_0)$ is:

Diagram 1.9.1.

C



Here $H_{\bar\varepsilon_j}=\bar d_j(E_{jj}-E_{j'j'})$ $(1\le j\le N)$, $E_i=E_{ii+1}-(-1)^{(\tilde p(i)\tilde p(i+1)+\tilde p(i+1))}g_ig_{i+1}E_{(i+1)'i'}$ $(1\le i\le N-1)$, $E_N=E_{NN'}$, $F_i=e\cdot\{(-1)^{\tilde p(i)}E_{i+1i}-(-1)^{\tilde p(i)\tilde p(i+1)}g_ig_{i+1}E_{i'(i+1)'}\}$ $(1\le i\le N-1)$, $F_N=e\cdot 2(-1)^{\tilde p(N)}E_{N'N}$.

Moreover we have:
If $p(\alpha_1)+\cdots+p(\alpha_{N-1})\equiv 1$, then

$$\theta=-\bar\varepsilon_1-\bar\varepsilon_2, F_\theta=e\cdot\{(-1)^{\tilde p(1)}E_{21'}-(-1)^{\tilde p(1)\tilde p(2)}g_2g_{1'}E_{12'}\},$$

$$E_\theta=E_{1'2}-(-1)^{\tilde p(1)\tilde p(2)}(-1)^{\tilde p(2)}g_{1'}g_2E_{2'1},$$

$$\psi=-2\bar\varepsilon_1, F_\psi=e\cdot 2(-1)^{\tilde p(1)}E_{11'}, E_\psi=E_{1'1}.$$

If $p(\alpha_1)+\cdots+p(\alpha_{N-1})\equiv 0$, then

$$\theta=-2\bar\varepsilon_1, F_\theta=e\cdot 2(-1)^{\tilde p(1)}E_{11'}, E_\theta=E_{1'1},$$

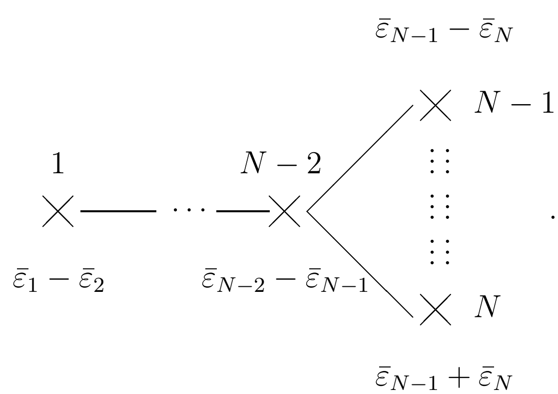
$$\psi=-\bar\varepsilon_1-\bar\varepsilon_2, F_\psi=e\cdot\{(-1)^{\tilde p(1)}E_{21'}+(-1)^{\tilde p(1)\tilde p(2)}g_2g_{1'}E_{12'}\},$$

$$E_\psi=E_{1'2}+(-1)^{\tilde p(1)\tilde p(2)}(-1)^{\tilde p(2)}g_{1'}g_2E_{2'1}.$$

1.10. D-type. If $\tilde N$ is even and $\tilde p(N)=0$, $e=\bar d_N$, then the Dynkin diagram of $(\mathcal{E}_0,\Pi_0,p_0)$ is:

Diagram 1.10.1.

D



Here $H_{\bar{\varepsilon}_j}=\bar{d}_j(E_{jj}-E_{j'j'})$ $(1\leq j\leq N)$, $E_i=E_{ii+1}-(-1)^{(\tilde{p}(i)\tilde{p}(i+1)+\tilde{p}(i+1))}g_ig_{i+1}E_{(i+1)'i'}$ $(1\leq i\leq N-1)$, $E_N=E_{N-1N'}-g_{N-1}g_{N'}E_{N(N-1)'}$,
$F_i=e\cdot\{(-1)^{\tilde{p}(i)}E_{i+1i}-(-1)^{\tilde{p}(i)\tilde{p}(i+1)}g_ig_{i+1}E_{i'(i+1)'}\}$ $(1\leq i\leq N-1)$,
$F_N=e\cdot\{(-1)^{\tilde{p}(N-1)}E_{N'N-1}-g_{(N-1)'}g_{N'}E_{(N-1)'N}\}$.

Moreover we have:
If $p(\alpha_1)+\cdots+p(\alpha_{N-1})\equiv 0$, then

$$\theta=-\bar{\varepsilon}_1-\bar{\varepsilon}_2, F_\theta=e\cdot\{(-1)^{\tilde{p}(1)}E_{21'}-(-1)^{\tilde{p}(1)\tilde{p}(2)}g_2g_{1'}E_{12'}\},$$

$$E_\theta=E_{1'2}-(-1)^{\tilde{p}(1)\tilde{p}(2)}(-1)^{\tilde{p}(2)}g_{1'}g_2E_{2'1},$$

$$\psi=-2\bar{\varepsilon}_1, F_\psi=e\cdot 2(-1)^{\tilde{p}(1)}E_{11'}, E_\psi=E_{1'1}.$$

If $p(\alpha_1)+\cdots+p(\alpha_{N-1})\equiv 1$, then

$$\theta=-2\bar{\varepsilon}_1, F_\theta=e\cdot 2(-1)^{\tilde{p}(1)}E_{11'}, E_\theta=E_{1'1}.$$

$$\psi=-\bar{\varepsilon}_1-\bar{\varepsilon}_2, F_\psi=e\cdot\{(-1)^{\tilde{p}(1)}E_{21'}+(-1)^{\tilde{p}(1)\tilde{p}(2)}g_2g_{1'}E_{12'}\},$$

$$E_\psi=E_{1'2}+(-1)^{\tilde{p}(1)\tilde{p}(2)}(-1)^{\tilde{p}(2)}g_{1'}g_2E_{2'1},$$

1.11. Proposition 1.11.1. $osp(\tilde{\mathcal{E}}_0,e)$ is the simple Lie superalgebra.

Proposition 1.11.2. *Let $(\mathcal{E}_0,\Pi_0,p_0)$ be the datum of $osp(\tilde{\mathcal{E}}_0,e)$ and $(\mathcal{E},\Pi,p)=(\mathcal{E}_0,\Pi_0,p_0)^{(I)}$. Put $\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)=\widehat{osp}(\tilde{\mathcal{E}}_0,e)^{(I)}$. There is an isomorphism: $j:\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)\to\mathcal{G}(\mathcal{E},\Pi,p)$ defined by letting $j(H\otimes 1+ac+bd)=H+aH_\delta+bH_{\Lambda_0}$ $(H\in\mathcal{H})$ and $j(E_\theta\otimes t)=E_0$, $j(F_\theta\otimes t^{-1})=F_0$.*

Proposition 1.11.3. *If $\sum_{i=1}^{\tilde{N}}(-1)^{\tilde{p}(i)}\neq 0$, then $sl(\tilde{\mathcal{E}}_0,e)_1^\Omega$ is the simple $osp(\tilde{\mathcal{E}}_0,e)$-module. If $(-1)^{\tilde{N}}=1$ and $\sum_{i=1}^N\bar{d}_i=0$, then the quotient $sl(\tilde{\mathcal{E}}_0,e)_1^\Omega/\mathcal{I}$ is the simple $osp(\tilde{\mathcal{E}}_0,e)$-module where*

$$\mathcal{I}=C\cdot\sum_{i=1}^N(E_{ii}+E_{i'i'}).$$

Proposition 1.11.4. *Assume $N\geq 3$ if $\alpha_N=2\bar{\varepsilon}_N$ and $p(\alpha_1)=1$. Let $(\mathcal{E}_0,\Pi_0,p_0)$ be the datum of $sl(\tilde{\mathcal{E}}_0,e)$ and $(\mathcal{E},\Pi,p)=(\mathcal{E}_0,\Pi_0,p_0)^{(\Omega)}$. Put $\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)=\hat{sl}(\tilde{\mathcal{E}}_0,e)^{(\Omega)}$. There is an epimorphism: $j:\hat{sl}(\tilde{\mathcal{E}}_0,e)^{(2)}\to\mathcal{G}(\mathcal{E},\Pi,p)$ defined by letting $j(H\otimes 1+ac+bd)=H+aH_\delta+bH_{\Lambda_0}$ $(H\in\mathcal{H})$ and $j(E_\psi\otimes t)=E_0$, $j(F_\psi\otimes t^{-1})=F_0$.*

If $\sum_{i=1}^N\bar{d}_i\neq 0$, then j is an isomorphism. If $(-1)^{\tilde{N}}=1$ and $\sum_{i=1}^N\bar{d}_i=0$, then

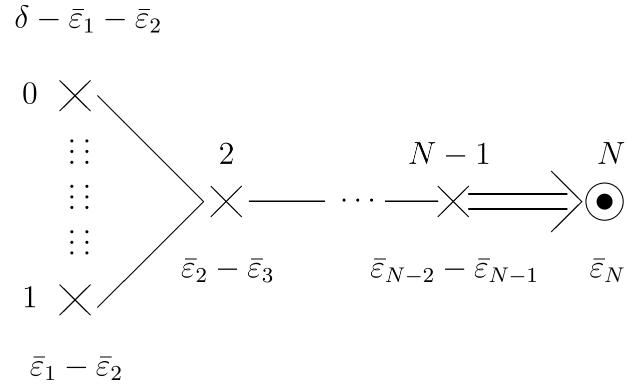
$$\ker j=\oplus_i\mathcal{I}\otimes t^{2i-1}.$$

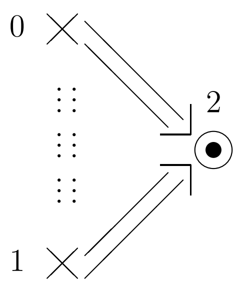
.

Example 1.11.1. The Dynkin diagrams of $(\mathcal{E}, \Pi, p)$ in Proposition 1.11.2 (resp. Proposition 1.11.4) are:

Diagram 1.11.1 ($N \geq 2$)

(DB)

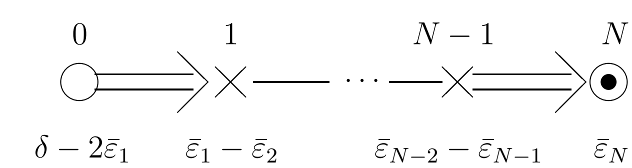


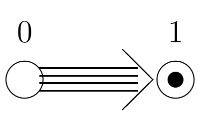


$(\sum_{i=1}^{N} p(\alpha_i) \equiv 0$ (resp. 1))

Diagram 1.11.2 ($N \geq 1$)

(CB)

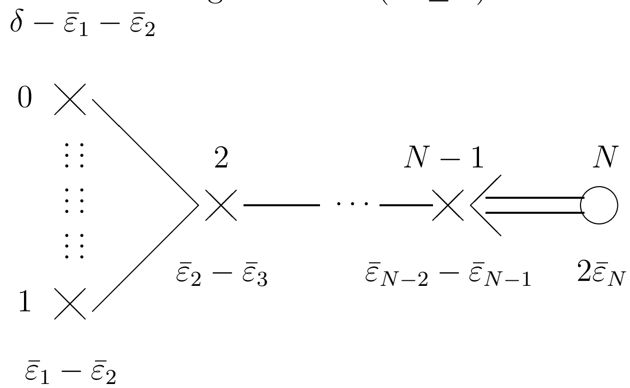




$\left(\sum_{i=1}^{N} p(\alpha_i) \equiv 1 \text{ (resp. } 0)\right)$

Diagram 1.11.3 ($N \geq 3$)

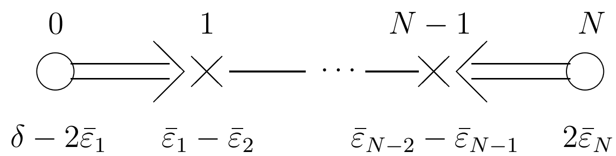
(DC)



$\left(\sum_{i=1}^{N-1} p(\alpha_i) \equiv 1 \text{ (resp. } 0)\right)$

Diagram 1.11.4 ($N \geq 3$)

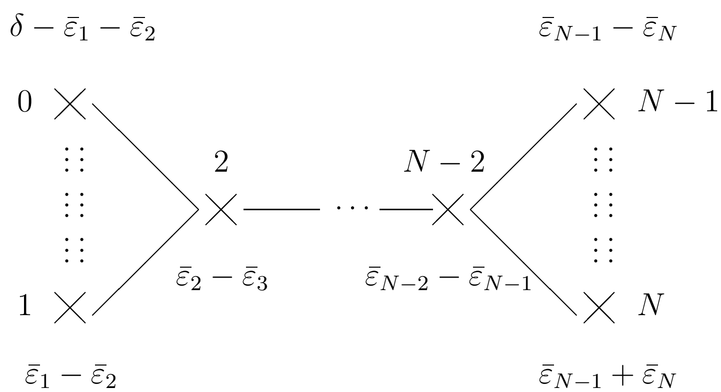
(CC)

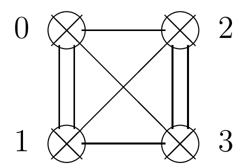


$\left(\sum_{i=1}^{N-1} p(\alpha_i) \equiv 0 \text{ (resp. } 1)\right)$

Diagram 1.11.5 ($N \geq 3$)

(DD)

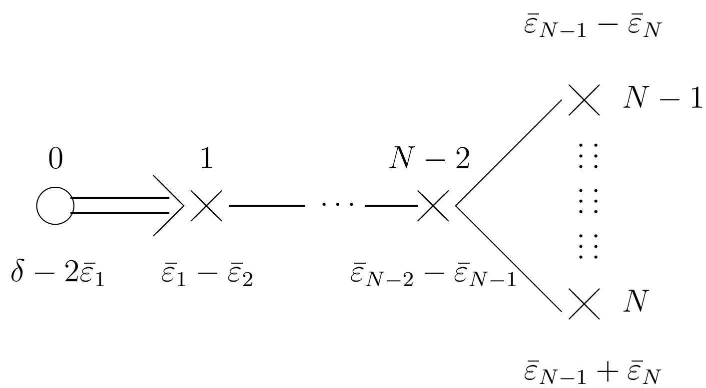




$$\left(\sum_{i=1}^{N-1} p(\alpha_i) \equiv 0 \text{ (resp. } 1)\right)$$

Diagram 1.11.6 ($N \geq 3$)

(CD)



$$\left(\sum_{i=1}^{N-1} p(\alpha_i) \equiv 1 \text{ (resp. } 0)\right)$$

1.12. Keep the notations in 1.10. However we denote the integer N in 1.10 by N_1. In 1.12, we let N denotes $N_1 - 1$. *We assumed* $\tilde{N} \geq 3$. *Then* $N \geq 2$. Let ω be an automorphism of $osp(\tilde{\mathcal{E}}_0, , \bar{d}_{N_1})$ of order 2 defined by:

$$\omega(X)_{ij} = X_{\hat{i}\hat{j}}$$

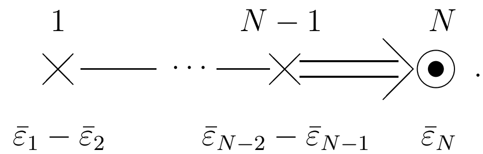
where

$$\hat{i} = \begin{cases} N' & \text{if } i = N_1, \\ N & \text{if } i = N_1', \\ i & \text{otherwise.} \end{cases}$$

Then we can identify $osp(\tilde{\mathcal{E}}_0, , \bar{d}_{N_1})_0^\omega$ with $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ of $(\mathcal{E}_0, \Pi_0, p_0)$ of the Dynkin diagram:

Diagram 1.12.1.

B



Here $H_{\bar{\varepsilon}_j} = \bar{d}_j(E_{jj} - E_{j'j'})$ $(1 \leq j \leq N-1)$, $E_i = E_{ii+1} - (-1)^{(\tilde{p}(i)\tilde{p}(i+1)+\tilde{p}(i+1))} g_i g_{i+1} E_{(i+1)'i'}$ $(1 \leq i \leq N-2)$, $E_{N_1-1} = E_{N_1-1N_1} + E_{N_1-1N_1'} - g_{N_1-1} g_{N_1'}(E_{N_1(N_1-1)'} + E_{N_1'(N_1-1)'})$,
$F_i = e \cdot \{(-1)^{\tilde{p}(i)} E_{i+1i} - (-1)^{\tilde{p}(i)\tilde{p}(i+1)} g_i g_{i+1} E_{i'(i+1)'}\}$ $(1 \leq i \leq N_1 - 2)$, $F_{N_1-1} = \frac{1}{2} \cdot e \cdot \{(-1)^{\tilde{p}(N_1-1)}(E_{N_1N_1-1} + E_{N_1'N_1-1}) - g_{(N_1-1)'} g_{N_1'}(E_{(N_1-1)'N} + E_{(N_1-1)'N_1'})\}$.

Here we introduce the lowest weight ψ of $osp(\tilde{\mathcal{E}}_0, e)_1^\omega$, a lowest weight vector E_ψ and a highest weight vector F_ψ such that $[E_\psi, F_\psi] = H_\psi$:

$$\psi = -\bar{\varepsilon}_1, F_\psi = \frac{1}{2} \cdot e \cdot \{E_{1N_1} + E_{1N_1'} - g_1 g_{N_1}(E_{N_1 1} + E_{N_1' 1})\},$$
$$E_\psi = E_{N1} + E_{N'1} - (-1)^{\tilde{p}(1)} g_N g_{1'}(E_{1'N} + E_{1'N'}).$$

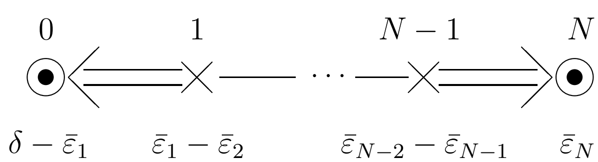
Proposition 1.12.2. *$osp(\tilde{\mathcal{E}}_0, e)_1^\omega$ is a simple $osp(\tilde{\mathcal{E}}_0, e)_0^\omega$-module.*

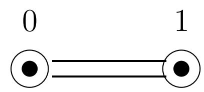
Proposition 1.12.3. *Let $(\mathcal{E}_0, \Pi_0, p_0)$ be the datum of $osp(\tilde{\mathcal{E}}_0, e)$ and $(\mathcal{E}, \Pi, p) = (\mathcal{E}_0, \Pi_0, p_0)^{(\omega)}$. Put $\bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p) = \widehat{osp}(\tilde{\mathcal{E}}_0, e)^{(\omega)}$. There is an isomorphism: $j : \bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p) \to \mathcal{G}(\mathcal{E}, \Pi, p)$ defined by letting $j(H \otimes 1 + ac + bd) = H + aH_\delta + bH_{\Lambda_0}$ $(H \in \mathcal{H})$ and $j(E_\psi \otimes t) = E_0$, $j(F_\psi \otimes t^{-1}) = F_0$.*

Example 1.12.1. If $N_1 \geq 3$, the Dynkin diagrams of $(\mathcal{E}, \Pi, p)$ in Proposition 1.12.1 are:

Diagram 1.12.2. $(N \geq 1)$

(BB)





$$(\textstyle\sum_{i=1}^{N} p(\alpha_i) \equiv 0)$$

1.13. Let $\tilde{\mathcal{E}}_0$ be the C-vector space in 1.5. Here we assume that $\tilde{N} = 2N+2$ $(N \geq 1)$ and shift the numbering of the basis as follows

$$\{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_N, \bar{\varepsilon}_{N+1}, \bar{\varepsilon}_{N'}, \ldots, \bar{\varepsilon}_{1'}\}$$

where $i' = 2N - i + 2$. We also assume:

$$\tilde{p}(0) = 1,\ \tilde{p}(N+1) = 0,\ \tilde{p}(i) = \tilde{p}(i')\ (1 \leq i \leq N),$$

and

$$g_i,\ g_{i'} \in \{\pm 1\}\ (1 \leq i \leq N) \quad g_i g_{i'} = (-1)^{\tilde{p}(i)}.$$

In 1.13, we denote this $\tilde{\mathcal{E}}_0$ by $\tilde{\mathcal{E}}_{01}$ and we again denote the subspace with the basis $\{\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_N, \bar{\varepsilon}_{N+1}, \bar{\varepsilon}_{N'}, \ldots, \bar{\varepsilon}_{1'}\}$ by $\tilde{\mathcal{E}}_0$. We denote an element $X \in sl(\tilde{\mathcal{E}}_{01}, e)$ by

$$X = \left(\begin{array}{c|c} \alpha & a_i \\ \hline b_i & X_{ij} \end{array}\right)$$

where the sizes of the matrices (α), (a_i), (b_i) and (X_{ij}) are 1×1, $1\times(2N+1)$, $(2N+1)\times 1$ and $(2N+1)\times(2N+1)$ respectively. Let Ξ be an automorphism of $sl(\tilde{\mathcal{E}}_{01}, e)$ of order 4 defined by:

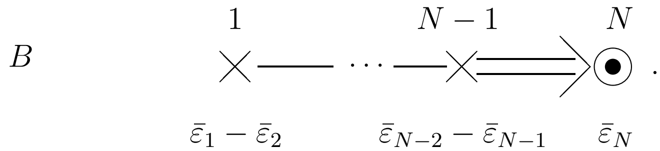
$$\Xi(X) = \left(\begin{array}{c|c} -\alpha & -\sqrt{-1} g_i b_{i'} \\ \hline -\sqrt{-1} g_{i'} a_{i'} & \Omega((X_{ij})) \end{array}\right).$$

Then $sl(\tilde{\mathcal{E}}_{01}, e)_0^{\Xi}$ consists of the matrices

$$X = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X_{ij} \end{array}\right)$$

where the $2N+1 \times 2N+1$-matrices (X_{ij}) form $osp(\tilde{\mathcal{E}}_0, e)$ whose Dynkin diagram is:

Diagram 1.13.1



Here we introduce the lowest weight ψ of $sl(\tilde{\mathcal{E}}_{01}, e)^{\Xi}$, a lowest weight vector $E_\psi \in sl(\tilde{\mathcal{E}}_{01}, e)^{\Xi}_{\bar{1}}$ and a highest weight vector $F_\psi \in sl(\tilde{\mathcal{E}}_{01}, e)^{\Xi}_{\bar{3}}$ such that $[E_\psi, F_\psi] = H_\psi$:

$$\psi = -\bar{\varepsilon}_1, F_\psi = e \cdot \{(-1)^{\tilde{p}(1)} E_{01'} + g_1 E_{1'0}\},$$

$$E_\psi = E_{1'0} - g_1 E_{01'}.$$

Proposition 1.13.1. $sl(\tilde{\mathcal{E}}_{01}, e)^{\Xi}_{\bar{1}}$ and $sl(\tilde{\mathcal{E}}_{1}, e)^{\Xi}_{\bar{3}}$ are the simple $osp(\tilde{\mathcal{E}}, e) = sl(\tilde{\mathcal{E}}_{01}, e)^{\Xi}_{\bar{0}}$-modules.

Proposition 1.13.2. As an $osp(\tilde{\mathcal{E}}_0, e)$-module,

$$sl(\tilde{\mathcal{E}}_1, e)^{\Xi}_{\bar{2}} \cong sl(\tilde{\mathcal{E}}, e)^{\Omega}_{\bar{1}} \oplus \mathcal{I}$$

where $\mathcal{I} = C \sum_{i=1}^{2N+1} \{(-1)^{\tilde{p}(i)} E_{00} + E_{ii}\}$.

Proposition 1.13.3. *Let* $(\mathcal{E}_0, \Pi_0, p_0)$ *be the datum of* $sl(\tilde{\mathcal{E}}_{01}, e)$ *and* $(\mathcal{E}, \Pi, p) = (\mathcal{E}_0, \Pi_0, p_0)^{(\Omega)}$. *Put* $\bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p) = \widehat{sl}(\tilde{\mathcal{E}}_{01}, e)^{(\Xi)}$. *There is an epimorphism:* $j : \hat{sl}(\tilde{\mathcal{E}}_0, e)^{(4)} \to \mathcal{G}(\mathcal{E}, \Pi, p)$ *defined by letting* $j(H \otimes 1 + ac + bd) = H + aH_\delta + bH_{\Lambda_0}$ $(H \in \mathcal{H})$ *and* $j(E_\psi \otimes t) = E_0$, $j(F_\psi \otimes t^{-1}) = F_0$.

If $\sum_{i=1}^{N} \bar{d}_i \neq 0$, *then* j *is an isomorphism. If* $(-1)^{\tilde{N}} = 1$ *and* $\sum_{i=1}^{N} \bar{d}_i = 0$, *then*

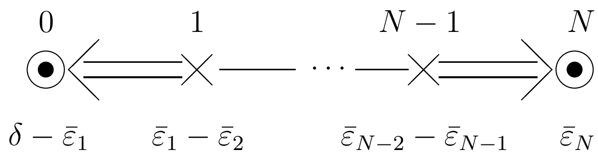
$$\ker j = \oplus_i \mathcal{I} \otimes t^{4i-2}.$$

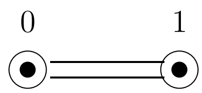
.

Example 1.13.1. The Dynkin diagrams of $(\mathcal{E}, \Pi, p)$ in Proposition 1.13.2 are:

Diagram 1.13.2. $(N \geq 1)$

(BB)





$(\sum_{i=1}^{N} p(\alpha_i) \equiv 1)$

1.14. We call the datum $(\mathcal{E}, \Pi, p)$ in the following table the one of *affine ABCD-type*. In the following table, a subscript of a name of a Dynkin diagram shows $\sum_{i=0}^{n} p(\alpha_i) \pmod 2$ of the corresponding superalgebra.

Name	Another name	Dynkin diagram	$\mathcal{G} = \bar{\bar{\mathcal{G}}}$
$A_{N-1}^{(1)}$	$\hat{sl}(\tilde{\mathcal{E}})^{(1)}$	$(AA)_0$	$\sum_{i=1}^{N} \bar{d}_i \neq 0$
$B_N^{(1)}$	$\widehat{osp}(\tilde{\mathcal{E}}_{odd})^{(1)}$	$(DB)_0\,(CB)_1$	*all*
$A_{2N}^{(2)}$	$\hat{sl}(\tilde{\mathcal{E}}_{odd})^{(2)}$	$(DB)_1\,(CB)_0$	*all*
$C_N^{(1)}\ D_N^{(1)}$	$\widehat{osp}(\tilde{\mathcal{E}}_{even})^{(1)}$	$(CC)_0\,(CD)_1\,(DD)_0\,(DC)_1$	*all*
$A_{2N-1}^{(2)}$	$\hat{sl}(\tilde{\mathcal{E}}_{even})^{(2)}$	$(CC)_1\,(CD)_0\,(DD)_1\,(DC)_0$	$\sum_{i=1}^{N} \bar{d}_i \neq 0$
$D_{N+1}^{(2)}$	$\widehat{osp}(\tilde{\mathcal{E}}_{even})^{(2)}$	$(BB)_0$	*all*
$A_{2N+1}^{(4)}$	$\hat{sl}(\tilde{\mathcal{E}})^{(4)}$	$(BB)_1$	$\sum_{i=1}^{N} \bar{d}_i \neq 0$

1.15. We call the data whose Dynkin diagrams are Diagram 5.1.4, Diagram 5.2.3 and Diagram 5.3.3 $D(2;1,x)^{(1)}$-type, $F_4^{(1)}$-type and $G_3^{(1)}$-type respectively. We call these *affine exceptional* type.

2. Affine Weyl type isomorphism

2.1. In this section, we introduce a family $\{L_i\}$ of isomorphisms between Lie superalgebras $\mathcal{G}(\mathcal{E}, \Pi, p)$ of $(\mathcal{E}, \Pi, p)$ of affine $ABCD$-type (see also [FSS]). The isomorphism L_i can be considered as a super-version of Weyl group action. However our isomorphisms change Dynkin diagrams. We shall also introduce other Lie superalgebras $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)$ of $(\mathcal{E}, \Pi, p)$ of affine ABCD-type.

We shall show that $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$'s are universal superalgebras satisfying: (i) $\mathcal{G}(\mathcal{E},\Pi,p)$ is a quotient of $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$, (ii) L_i can be lifted to isomorphisms of $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$'s. Finally we shall show that $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)=\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)$. We have already introduced $\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)$ in a concrete way in §1 for $(\mathcal{E},\Pi,p)$ of affine $ABCD$-type. We have known that $\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)\neq\mathcal{G}(\mathcal{E},\Pi,p)$ if and only if $\mathcal{G}(\mathcal{E},\Pi,p)=\hat{sl}(\tilde{\mathcal{E}})^{(1)}$, $\hat{sl}(\tilde{\mathcal{E}}_{even})^{(2)}$, $\hat{sl}(\tilde{\mathcal{E}})^{(4)}$ and $\sum_{i=1}^{N}\bar{d}_i=0$.

Our idea using Weyl-group-type isomorphisms relates to [LS].

2.2. Let

$$\breve{E}_{ij}(k)=[...[[E_j,\underbrace{E_i],E_i],\ldots,E_i]}_{k-\text{ times}},\quad \breve{F}_{ij}(k)=[...[[F_j,\underbrace{F_i],F_i],\ldots,F_i]}_{k-\text{ times}}.$$

The calculations of the following lemma are useful.

Lemma 2.2.1. (i) If $i\neq j$, then $[[E_j,E_i],F_i]=-(\alpha_i,\alpha_j)E_j$, $[E_i,[F_j,F_i]]=(-1)^{p(\alpha_i)p(\alpha_j)}(\alpha_i,\alpha_j)F_j$ $[[E_j,E_i],[F_j,F_i]]=(-1)^{p(\alpha_i)p(\alpha_j)}(\alpha_i,\alpha_j)H_{\alpha_i+\alpha_j}$.
(ii) For the i-th simple root $\alpha_i\in\Pi$ satisfying $(\alpha_i,\alpha_i)\neq 0$, let $a_{ij}=\dfrac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$. Assume that a_{ij} is an even integer if $p(\alpha_i)=1$. Put

$$<k;-a_{ij}>=\begin{cases} k(-a_{ij}-k+1) & \text{if } p(\alpha_i)=0,\\ k & \text{if } p(\alpha_i)=1 \text{ and } k \text{ is even},\\ -a_{ij}-k+1 & \text{if } p(\alpha_i)=1 \text{ and } k \text{ is odd}.\end{cases}$$

We put $<k;-a_{ij}>!=\prod_{r=1}^{k}<r;-a_{ij}>$. Then

$$[E_i,\breve{F}_{ij}(k)]=-(-1)^{p(\alpha_i)p(\alpha_j)}<k;-a_{ij}>\breve{F}_{ij}(k-1),$$

$$[\breve{E}_{ij}(k),F_i]=(-1)^{(k-1)p(\alpha_j)}<k;-a_{ij}>\breve{E}_{ij}(k-1),$$

$$[\breve{E}_{ij}(k),\breve{F}_{ij}(k)]=(-1)^{k}(-1)^{p(\alpha_i)p(\alpha_j)}<k;-a_{ij}>!\,H_{k\alpha_i+\alpha_j}.$$

Lemma 2.2.2. *Let $\mathcal{G}=\mathcal{G}(\mathcal{E},\Pi,p)$ be a Kac-Moody superalgebra with the triangular decomposition $\mathcal{G}=\mathcal{N}^+\otimes\mathcal{H}\otimes\mathcal{N}^-$. If $X\in\mathcal{N}^+$ (resp. $Y\in\mathcal{N}^-$) satisfies $[X,F_k]=0$ (resp. $[E_k,Y]=0$) for any k, then $X=0$ (resp. $Y=0$) in $\mathcal{G}$.*

Proof. We can assume that X is in an root space. Let $r_+(X)$ be the ideal of $\mathcal{N}^+$ generated by X. Then $r_+(X)$ is an ideal of $\mathcal{G}$ such that $r_+(X)\cap\mathcal{H}=0$. Hence $X=0$.

Q.E.D.

As an immediate consequence of Lemma 2.2.1 and Lemma 2.2.2, we have:

Lemma 2.2.3 *(i) For the i-th simple root $\alpha_i \in \Pi$ satisfying $(\alpha_i, \alpha_i) \neq 0$, let $a_{ij} = 2(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i)$. Assume that a_{ij} is an even integer if $p(\alpha_i) = 1$. Then $\breve{E}_{ij}(-a_{ij}) = 0$, $\breve{F}_{ij}(-a_{ij}) = 0$ in $\mathcal{G}$.*
(ii) If $(\alpha_i, \alpha_i) = 0$, then $[E_i, E_i] = [F_i, F_i] = 0$ in $\mathcal{G}$.
Proof. Direct calculations.

Proposition 2.2.1. *Let $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$ be a Kac-Moody superalgebra such that an i-th simple root $\alpha_i \in \Pi$ satisfies $(\alpha_i, \alpha_i) = 0$. Let $\Pi' = \{\alpha'_1, \ldots \alpha'_n\}$ $= \{-\alpha_i, \alpha_j + \alpha_i \;\; (j \neq i, (\alpha_i, \alpha_j) \neq 0), \;\; \alpha_j \, (j \neq i, (\alpha_i, \alpha_j) = 0)\}$. Put $\mathcal{G}' = \mathcal{G}(\mathcal{E}, \Pi', p)$. Then there are isomorphisms $\phi : \mathcal{G} \to \mathcal{G}'$ such that*

$$\phi(H_\gamma) = H_\gamma. \tag{2.2.1}$$

(In particular,

$$\phi(H_\gamma) = \begin{cases} -H_{\alpha'_i} & \text{if } \gamma = \alpha_i, \\ H_{\alpha'_i + \alpha'_j} & \text{if } \gamma = \alpha_j \text{ and } (\alpha_i, \alpha_j) \neq 0, \\ H_{\alpha'_j} & \text{if } \gamma = \alpha_j \text{ and } (\alpha_i, \alpha_j) = 0. \end{cases}$$

)

$$\phi(E_i) = -(-1)^{p(\alpha_i)} F_i, \quad \phi(F_i) = -E_i, \tag{2.2.2}$$

$$\phi(E_j) = -\frac{(-1)^{p(\alpha_i)p(\alpha_j)}}{(\alpha_i, \alpha_j)}[E_j, E_i], \quad \phi(F_j) = -[F_j, F_i], \; (\; (\alpha_i, \alpha_j) \neq 0 \;) \tag{2.2.3}$$

$$\phi(E_j) = E_j, \quad \phi(F_j) = F_j \quad (\; i \neq j, \; (\alpha_i, \alpha_j) = 0 \;) \tag{2.2.4}$$

Proof. Let $\mathcal{H}'$ be the Cartan subalgebra of $\mathcal{G}'$. Denote the right hand sides of (2.2.1-4) by H'_γ, E'_j and F'_j. We can show that there is an epimorphism $y : \tilde{\mathcal{G}}(\mathcal{E}, \Pi, p) \to \mathcal{G}'$. such that $y(H_\gamma) = H'_\gamma$, $y(E_j) = E'_j$ and $y(F_j) = F'_j$. For example, by Lemma 2.2.3 (ii), we can show that the elements satisfy (1.2.1-3). Clearly $y_{|\mathcal{H}} : \mathcal{H} \to \mathcal{H}'$ is isomorphism. Hence there exists an epimorphism $\phi_1 : \mathcal{G}' \to \mathcal{G}$ such that $\phi_1(H'_\gamma) = H_\gamma$, $\phi_1(E'_j) = E_j$ and $\phi_1(F'_j) = F_j$. Since $\phi_{1|\mathcal{H}'}$ is injective, ϕ_1 is isomorphism. ϕ_1 is nothing else but ϕ^{-1}.

Q.E.D.

Keep the notations in the statement of Proposition 2.2.1. We still assume $(\alpha_i, \alpha_i) = 0$. For the Dynkin diagram Γ of $(\mathcal{E}, \Pi, p)$, we denote the Dynkin

diagram of $(\mathcal{E},\Pi',p)$ by $\Gamma^{<i>}$. Similarly to the proof of Proposition 2.2.1, we have following Propositions 2.2.2-3.

Proposition 2.2.2. *Let $\mathcal{G}=\mathcal{G}(\mathcal{E},\Pi,p)$ be a Kac-Moody superalgebra such that an i-th simple root $\alpha_i\in\Pi$ satisfies $(\alpha_i,\alpha_i)\neq 0$. Let $d_i=(\alpha_i,\alpha_i)/2$. Let*

$$(-a_{ij})_s! = \begin{cases} (-a_{ij})! & \text{if } p(\alpha_i)=0,\\ (\frac{-a_{ij}}{2})!2^{\frac{-a_{ij}}{2}} & \text{if } p(\alpha_i)=1, \end{cases}$$

Assume that a_{ij} is an even integer if $p(\alpha_i)=1$. Then there is an isomorphisms $\phi:\mathcal{G}\to\mathcal{G}$ such that

$$\phi(H_\gamma)=H_{\gamma-\frac{2(\gamma,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i} \tag{2.2.7}$$

$$\phi(E_i)=-(-1)^{p(\alpha_i)}F_i,\quad \phi(F_i)=-E_j, \tag{2.2.8}$$

$$\phi(E_j)=\frac{1}{(-a_{ij})_s!d_i^{-a_{ij}}}\breve{E}_{ij}(-a_{ij}), \tag{2.2.9}$$

$$\phi(F_j)=(-1)^{-a_{ij}}\frac{1}{(-a_{ij})_s!}\breve{F}_{ij}(-a_{ij}). \tag{2.2.10}$$

Proposition 2.2.3. *For $\Pi=\{\alpha_1,\ldots,\alpha_n\}$, let $a:\{1,\ldots,n\}\to\{1,\ldots,n\}$ be a bijective map such that $(\alpha_{a(i)},\alpha_{a(j)})=(\alpha_i,\alpha_j)$ $(1\leq i,\,j\leq n)$. Then there is an isomorphisms $\phi:\mathcal{G}\to\mathcal{G}$ such that*

$$H_{\alpha_i}=H_{\alpha_{a(i)}}, \tag{2.2.11}$$

$$\phi(E_i)=E_{a(i)},\phi(F_i)=F_{a(i)}. \tag{2.2.12}$$

2.3. Here we fix a positive integer N. Let Θ_N be the set of affine $ABCD$-type Dynkin diagrams Γ satisfying that the number of the dots of Γ is $N-1$ if Γ is of the affine A-type, N otherwise. Let $D(\Theta_N)$ be the set of the data $(\mathcal{E},\Pi=\{\alpha_0,\alpha_1,\ldots,\alpha_n\},p)$ whose Dynkin diagrams belong to Θ_N. Let $\Gamma\in\Theta_N$ and $(\mathcal{E},\Pi,p)\in D(\Theta_N)$. For $0\leq i\leq n$, we define $\Gamma^{\sigma(i)}=\Gamma'\in\Theta_N$ and $(\mathcal{E}^{\sigma(i)}=\oplus_{i=1}^N C\bar{\varepsilon}'_i\oplus C\delta\oplus C\Lambda_0,\,\Pi^{\sigma(i)},p^{\sigma(i)})\in D(\Theta_N)$ by following (i)-(iii): (We put $\bar{d}'_i=(\bar{\varepsilon}'_i,\bar{\varepsilon}'_i)$.)

(i) If $1\leq i\leq N-1$ and $(\alpha_i,\alpha_i)=0$, then $(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})$ satisfies that $\Gamma^{\sigma(i)}=\Gamma^{<i>}$ (read the sentences before Proposition 2.2.2) and $\bar{d}'_i=\bar{d}_{i+1}$, $\bar{d}'_{i+1}=\bar{d}_i$, $\bar{d}'_j=\bar{d}_j$ $(j\neq i,i+1)$.

(ii) If Γ is affine A type and $i=0$, then $(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})$ satisfies that $\Gamma^{\sigma(i)}=\Gamma^{<i>}$ and $\bar{d}'_1=\bar{d}_N$, $\bar{d}'_N=\bar{d}_1$ and $\bar{d}'_j=\bar{d}_j$ $(j\neq 1,N)$.

(iii) Otherwise we put $\Gamma^{\sigma(i)} = \Gamma$ and $(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)}) = (\mathcal{E}, \Pi, p)$.

2.4. For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$, let $(\mathcal{E}^\dagger,\ \Pi^\dagger = \{\alpha_0^\dagger, \ldots, \alpha_n^\dagger\},\ p^\dagger) \in D(\Theta_N)$ be an another datum satisfying:

(i) $\mathcal{E}^\dagger = \mathcal{E}$,

(ii) For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$, the type of the Dynkin diagram $\Gamma^\dagger$ of $(\mathcal{E}^\dagger, \Pi^\dagger, p^\dagger) \in D(\Theta_N)$ is:

$$\begin{cases} (AA) & \text{if } \Gamma \text{ is type } (AA), \\ (BB) & \text{if } \Gamma \text{ is type } (BB), \\ (CB) & \text{if } \Gamma \text{ is type } (CB) \text{ or } (DB), \\ (CC) & \text{if } \Gamma \text{ is type } (CC), (CD), (DC) \text{ or } (DD). \end{cases}$$

We shall not need $p^\dagger$. So we merely denote $(\mathcal{E}^\dagger, \Pi^\dagger, p^\dagger)$ by $(\mathcal{E}^\dagger, \Pi^\dagger)$.

Remark 2.4.1. We can easily see that two $\Gamma^\dagger$'s defined for $(\mathcal{E}, \Pi, p)$ and $(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$ are same.

Lemma 2.4.1. $P_+ = Z_+\alpha_0 \oplus \ldots \oplus Z_+\alpha_n \subset P_+^\dagger = Z_+\alpha_0^\dagger \oplus \ldots \oplus Z_+\alpha_n^\dagger$.

2.5. On $\mathcal{E} = \oplus_{i=1}^N C\bar{\varepsilon}_i \oplus C\delta \oplus C\Lambda_0$, we define another symmetric form $((\ ,\ )) : \mathcal{E} \times \mathcal{E} \to C$ by

$$((\bar{\varepsilon}_i, \bar{\varepsilon}_i)) = \delta_{ij}, ((\bar{\varepsilon}_i, \delta)) = 0, ((\delta, \delta)) = 0, ((\delta, \Lambda_0)) = 1, ((\Lambda_0, \Lambda_0)) = 1.$$

For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$ and $0 \le i \le n$, define $I^{\sigma(i)} : \mathcal{E} = \oplus_{i=1}^N C\bar{\varepsilon}_i \oplus C\delta \oplus C\Lambda_0 \to \mathcal{E}^{\sigma(i)} = \oplus_{i=1}^N C\bar{\varepsilon}_i' \oplus C\delta \oplus C\Lambda_0$ by $I^{\sigma(i)}(\bar{\varepsilon}_i) = \bar{\varepsilon}_i'$, $I^{\sigma(i)}(\delta) = \delta$ and $I^{\sigma(i)}(\Lambda_0) = \Lambda_0$. We also define a linear map $\sigma(i) : \mathcal{E} \to \mathcal{E}^{\sigma(i)}$ by

$$\sigma(i)\,(v) = I^{\sigma(i)}(v - \frac{2((v, \alpha_i^\dagger))}{((\alpha_i^\dagger, \alpha_i^\dagger))}\alpha_i^\dagger).$$

As an easy consequence from Propositions 2.2.1-3, we have:

Theorem 2.5.1. *For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$ and $0 \le i \le n$, Let $\mathcal{H} \oplus \oplus_{\alpha \in \Phi^{\sigma(i)}} \mathcal{G}_\alpha^{\sigma(i)}$ be the root space decomposition of $\mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$.*

(i) There is an isomorphism $L_i : \mathcal{G}(\mathcal{E}, \Pi, p) \to \mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$ such that:

$$L_i(H_\gamma) = H_{\sigma(i)(\gamma)}\,(\gamma \in \mathcal{E}). \tag{2.5.1}$$

In particular, L_i satisfies:

$$L_i(\mathcal{G}_\alpha) = \mathcal{G}^{\sigma(i)}_{\sigma(i)(\alpha)} \quad (\alpha \in \Phi). \tag{2.5.2}$$

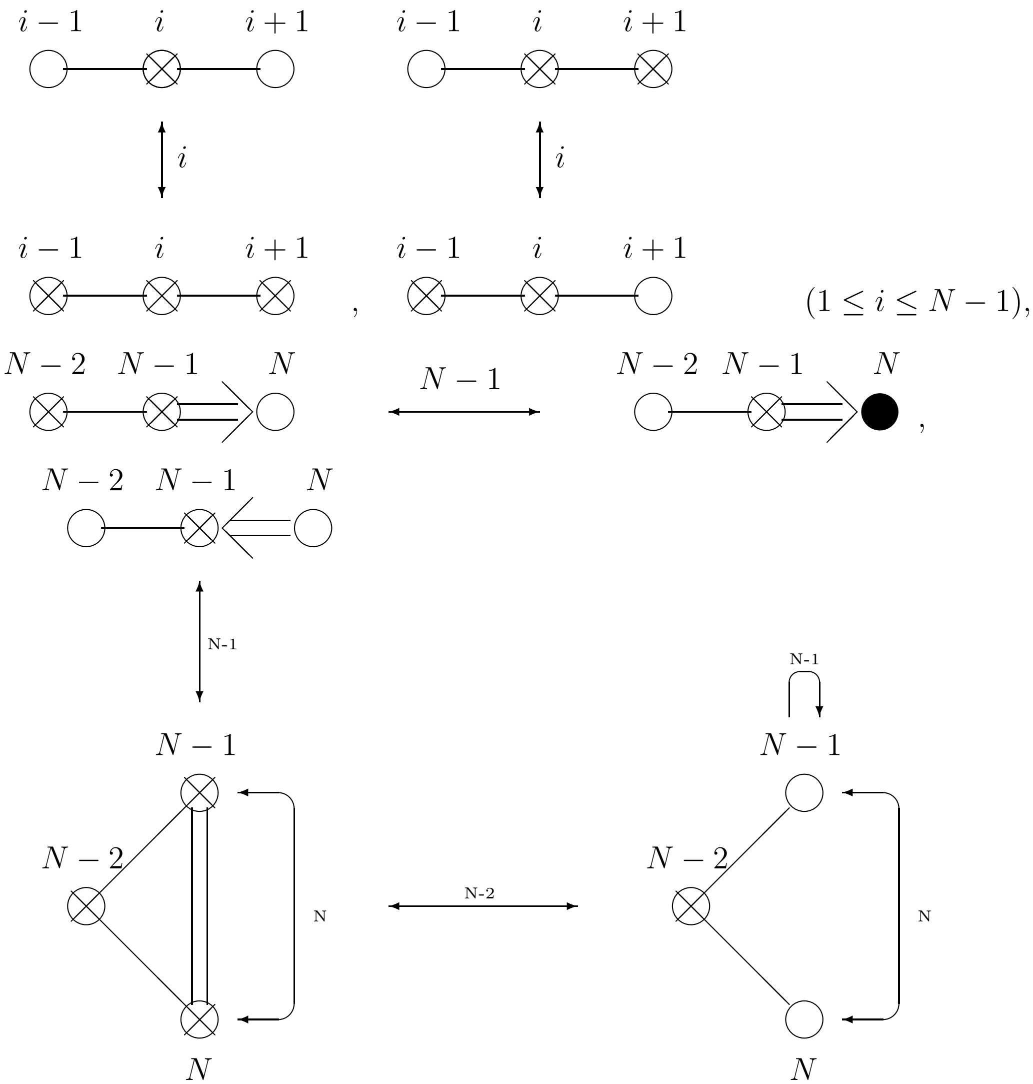
(ii) Let L'_i and L_i be two isomorphisms satisfying (2.5.1) Then there exist $a_j \in C^*$ such that

$$L'_i(E_j) = a_j L_i(E_j), \ L'_i(F_j) = a_j^{-1} L_i(F_j). \tag{2.5.3}$$

Proof. (i) We can choose one of ϕ's in Propositions 2.2.1-3 as L_i. It is obvious L_i satisfies (2.5.1-2).

(ii) By (2.5.2) and the fact of $\dim \mathcal{G}_{\alpha_j} = 1$, $\dim \mathcal{G}^{\sigma(i)}_{\sigma(i)(\alpha_j)} = 1$. By the fact of $[L_i(E_j), L_i(F_j)] = H_{\sigma(i)(\alpha_j)} = [L'_i(E_j), L'_i(F_j)]$, we get (2.5.3).

Remark 2.5.1. For example, $\sigma(i)$ and L_i move as follows:



Proposition 2.5.1. For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$, let $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$. Then $\dim \mathcal{G}_\alpha = 1$ if $\alpha \in \Phi \setminus Z\delta$.

Proof. We consider the case of $\alpha \in \Phi_+ \setminus Z_+\delta$. We use an induction on the height with respect to $\Pi^\dagger$. By $((\alpha, \alpha)) > 0$, $((\alpha, \alpha_i)) > 0$ for some i. If

$\alpha \notin Z\alpha_i$ then the height of $\sigma(i)(\alpha)$ is smaller than the height of α. Thus by $\sigma(i)$'s, we can get a path from α to $r\alpha_i \in \Pi'$ of another datum $(\mathcal{E}', \Pi', p')$ where $r \in \{1, 2\}$ if $p(\alpha_i) = 1$ and $(\alpha_i, \alpha_i) \neq 0$, $r = 1$ otherwise. On the other hand, $\dim \mathcal{G}'_{r\alpha_j} = 1$ by Lemma 1.2.1.

Q.E.D.

2.6. Let $D(\Theta_N)_{(XY)}$ denote $\{(\mathcal{E}, \Pi, p) \in \Theta_N | (\mathcal{E}, \Pi, p) \ is \ (XY) - type\}$. Then $\{\sigma(i) | 1 \leq i \leq n\}$ preserve $D(\Theta_N)_1 = D(\Theta_N)_{(AA)}$, $D(\Theta_N)_2 = D(\Theta_N)_{(BB)}$, $D(\Theta_N)_3 = D(\Theta_N)_{(CB)} \cup D(\Theta_N)_{(DB)}$ or $D(\Theta_N)_4 = D(\Theta_N)_{(CC)} \cup D(\Theta_N)_{(CD)} \cup D(\Theta_N)_{(DC)} \cup D(\Theta_N)_{(DD)}$. Let $W = W(i)$ denote a group generated by $\{\sigma(i) | 1 \leq i \leq n\}$ acting on $D(\Theta_N)_i$. Then $W(i)$ is isomorphic to the affine Weyl group of the corresponding $(\mathcal{E}^\dagger, \Pi^\dagger)$.

Let $D(\Theta_N)_i = \cup_{j=1}^{a_i} D(\Theta_N)_{ij}$ be the orbit decomposition. If a datum $(\mathcal{E}, \Pi, p)$ associated to a Dynkin diagram $\Gamma \in \Theta_N$ belongs to $D(\Theta_N)_{ij}$, then we denote $D(\Theta_N)_{ij}$ by $D(\Theta_N)[\Gamma]$.

2.7. Proposition 2.7.1. *Fix an orbit $D(\Theta_N)[\Gamma]$. Fix an isomorphism $L_i : \mathcal{G}(\mathcal{E}, \Pi, p) \to \mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$ for each $(\mathcal{E}, \Pi, p)$ and $\sigma(i)$. Then there exists a unique family $\{\bar{\mathcal{G}}(\mathcal{E}, \Pi, p) \,|(\mathcal{E}, \Pi, p) \in D(\Theta_N)[\Gamma]\}$ of Lie superalgebras satisfying following (1), (2) and (3).*

(1) For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)[\Gamma]$, there is a sequence of epimorphisms

$$\begin{array}{ccccc} \tilde{\mathcal{G}}(\mathcal{E}, \Pi, p) & \overset{\tilde{\Psi}(\mathcal{E},\Pi,p)}{\twoheadrightarrow} & \bar{\mathcal{G}}(\mathcal{E}, \Pi, p) & \overset{\Psi(\mathcal{E},\Pi,p)}{\twoheadrightarrow} & \mathcal{G}(\mathcal{E}, \Pi, p) \\ (H_\gamma, E_i, F_i & \longrightarrow & H_\gamma, E_i, F_i & \longrightarrow & H_\gamma, E_i, F_i). \end{array}$$

(2) For the isomorphism $L_i : \mathcal{G}(\mathcal{E}, \Pi, p) \to \mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$, there is an isomorphism $\bar{L}_i : \bar{\mathcal{G}}(\mathcal{E}, \Pi, p) \to \bar{\mathcal{G}}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$ satisfying the following commuting diagram:

$$\begin{tikzcd} \bar{\mathcal{G}}(\mathcal{E}, \Pi, p) \arrow[r, "\bar{L}_i"] \arrow[d, "\Psi(\mathcal{E},\Pi,p)"'] \arrow[dr, phantom, "\circlearrowright"] & \bar{\mathcal{G}}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)}) \arrow[d, "\Psi(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})"] \\ \mathcal{G}(\mathcal{E}, \Pi, p) \arrow[r, "L_i"] & \mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)}) \end{tikzcd}$$

(3) If there is a family $\{\bar{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi,p)\ |(\mathcal{E},\Pi,p)\in D(\Theta_N)[\Gamma]\}$ of Lie superalgebras satisfying (1) and (2), then there are epimorphisms

$$\bar{\Psi}^{(\lambda)}(\mathcal{E},\Pi,p):\bar{\mathcal{G}}(\mathcal{E},\Pi,p)\twoheadrightarrow\tilde{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi,p)\qquad((\mathcal{E},\Pi,p)\in D(\Theta_N)[\Gamma])$$

satisfying following commutative diagram

Diagram 2.7.1.

$$\begin{tikzcd}
\bar{\mathcal{G}}(\mathcal{E},\Pi,p) \arrow[rr, "\bar{L}_i"] \arrow[dr, "\bar{\Psi}^{(\lambda)}(\mathcal{E},\Pi,p)"] \arrow[dd, "\Psi(\mathcal{E},\Pi,p)"'] & & \bar{\mathcal{G}}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)}) \arrow[dr, "\bar{\Psi}^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})"] \arrow[dd, "\Psi(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})"'] & \\
& \bar{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi,p) \arrow[rr, "\bar{L}_i^{(\lambda)}"] \arrow[dl, "\Psi^{(\lambda)}(\mathcal{E},\Pi,p)"] & & \bar{\mathcal{G}}^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)}) \arrow[dl, "\Psi^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})"] \\
\mathcal{G}(\mathcal{E},\Pi,p) \arrow[rr, "L_i"] & & \mathcal{G}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)}) &
\end{tikzcd}$$

(ii) The set $\{\bar{\mathcal{G}}(\mathcal{E},\Pi,p)\ |(\mathcal{E},\Pi,p)\in D(\Theta_N)[\Gamma]\}$ does not depend on the choice of $\{L_i\}$.

Proof. (i) Let $\{\,C^{(\lambda)}=\{\bar{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi,p)|(\mathcal{E},\Pi,p)\in D(\Theta_N)[\Gamma]\}\,\}_{(\lambda)\in(\Lambda)}$ be the family of the families $C^{(\lambda)}$ ($(\lambda)\in(\Lambda)$) satisfying (1) and (2). Let $\tilde{\Psi}^{(\lambda)}(\mathcal{E},\Pi,p):\tilde{\mathcal{G}}(\mathcal{E},\Pi,p)\to\bar{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi,p)$ be the epimorphism in (1) for $(\lambda)\in(\Lambda)$. Let $\bar{r}^{(\lambda)}(\mathcal{E},\Pi,p)=\ker\tilde{\bar{\Psi}}^{(\lambda)}(\mathcal{E},\Pi,p)$ and $\bar{r}(\mathcal{E},\Pi,p)=\cap_{(\lambda)\in(\Lambda)}\bar{r}^{(\lambda)}(\mathcal{E},\Pi,p)$. Put $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)=\tilde{\mathcal{G}}(\mathcal{E},\Pi,p)/\bar{r}(\mathcal{E},\Pi,p)$. Let $[L_i(E_j)]$, $[L_i(F_j)]\in\bar{\mathcal{G}}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})_{\sigma(i)(\alpha_j)}$ be representatives of $L_i(E_j)$, $L_i(F_j)$. Similarly to Proposition 2.5.1, we can show $\dim\bar{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi,p)_\alpha=1$ if $\alpha\in\Phi\setminus Z\delta$. Then $\bar{r}^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})_{\sigma(i)(\alpha_j)}$ $=r(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})_{\sigma(i)(\alpha_j)}$. Hence $[L_i(E_j)]\equiv\bar{L}_i^{(\lambda)}(E_j)$, $[L_i(F_j)]\equiv\bar{L}_i^{(\lambda)}(F_j)$ (mod $\bar{r}^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})$).
Hence, in $\bar{\mathcal{G}}^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})$, the elements $H_{\sigma(i)(\gamma)}$, $[L_i(E_j)]$ and $[L_i(F_j)]$ satisfy (1.2.1-3) for $(\mathcal{E},\Pi,p)$ whence, even in $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$, the elements satisfy

(1.2.1-3). Hence there is a morphism $\tilde{\bar{L}}_i : \tilde{\mathcal{G}}(\mathcal{E},\Pi,p) \to \bar{\mathcal{G}}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})$ such that

$$\bar{\Psi}^{(\lambda)}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)}) \circ \tilde{\bar{L}}_i = \bar{L}_i^{(\lambda)} \circ \tilde{\Psi}^{(\lambda)}(\mathcal{E},\Pi,p). \tag{2.7.1}$$

Therefore there exists $\bar{L}_i : \bar{\mathcal{G}}(\mathcal{E},\Pi,p) \to \bar{\mathcal{G}}(\mathcal{E}^{\sigma(i)},\Pi^{\sigma(i)},p^{\sigma(i)})$ such that $\tilde{\bar{L}}_i = \bar{L}_i \circ \tilde{\Psi}(\mathcal{E},\Pi,p)$. Clearly $\bar{L}_i$ satisfying the commutative diagram of (2). Using Lemma 2.2.1, we can see that $\bar{L}_i \circ \bar{L}_i : \bar{\mathcal{G}}(\mathcal{E},\Pi,p) \to \bar{\mathcal{G}}(\mathcal{E},\Pi,p)$ satisfies that $\bar{L}_i \circ \bar{L}_i(H_\gamma) = (H_\gamma)$, and $\bar{L}_i \circ \bar{L}_i(E_j)$, $\bar{L}_i \circ \bar{L}_i(F_j)$ are nonzero scalar multiples of E_j, F_j respectively. Hence $\bar{L}_i$ is an isomorphism.
(ii) By theorem 2.5.1, if L_i' is another L_i, then $L_i'(E_j) = a_j L_i(E_j)$, $L_i'(F_j) = a_j^{-1} L_i(F_j)$ for some $a_j \in C \setminus \{0\}$ $(0 \le j \le n)$. Let $\phi_a : \mathcal{G}(\mathcal{E},\Pi,p) \to \mathcal{G}(\mathcal{E},\Pi,p)$ be an isomorphism defined by $\phi_a(E_j) = a_j E_j$, $\phi_a(F_j) = a_j^{-1} F_j$, $\phi_a(H) = H$. Then $L_i' = L_i \circ \phi_a$. The ideal $\bar{r}(\mathcal{E},\Pi,p)$ in the proof of (i) satisfies $\mathcal{H} \cap \bar{r}(\mathcal{E},\Pi,p) = 0$. Then $\bar{r}(\mathcal{E},\Pi,p)$ is the homogeneous ideal. Then we can also define an isomorphism $\bar{\phi}_a : \bar{\mathcal{G}}(\mathcal{E},\Pi,p) \to \bar{\mathcal{G}}(\mathcal{E},\Pi,p)$ similar to ϕ_a. Denote $\bar{L}_i$ defined for L_i' by $\bar{L}_i'$. By the universality of $\bar{L}_i$, it follows that $\bar{L}_i' = \bar{L}_i \circ \bar{\phi}_a$. In particular, $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$ is determined independently of the choice of L_i.

Q.E.D.

Lemma 2.7.1. *Let $(\mathcal{E},\Pi,p) \in D(\Theta_N)$ and $\bar{\mathcal{G}} = \bar{\mathcal{G}}(\mathcal{E},\Pi,p)$. Let $\bar{\mathcal{N}}^+$, $\bar{\mathcal{N}}^-$ be the subalgebras of $\bar{\mathcal{G}}$ generated by E_i, F_i respectively. Then we have the triangular decomposition*

$$\bar{\mathcal{G}} = \bar{\mathcal{N}}^+ \oplus \mathcal{H} \oplus \bar{\mathcal{N}}^-.$$

Here $\mathcal{H}$ can be identified with $\mathcal{H}$ of $\mathcal{G}$. We have $\bar{\mathcal{N}}^+ \cong \bar{\mathcal{N}}^-$ $(E_i \leftrightarrow F_i)$.
Proof. The triangular decomposition is clear because $\bar{r} \cap \mathcal{H} = 0$. Let $\bar{r}$ be the ideal $\bar{r}(\mathcal{E},\Pi,p)$ of $\tilde{\mathcal{G}}(\mathcal{E},\Pi,p)$ in the proof of Proposition 2.7.1. Let $\bar{r}_\pm = \bar{r} \cap \widetilde{\mathcal{N}}^\pm$. Let $\bar{r}_-^1$ (resp. $\bar{r}_+^1$) be the ideal defined as the image of $\bar{r}_+$ of the map $\widetilde{\mathcal{N}}^+ \to \widetilde{\mathcal{N}}^-$ $(E_i \to F_i)$ (resp. $\widetilde{\mathcal{N}}^- \to \widetilde{\mathcal{N}}^+$ $(F_i \to E_i)$). Put $\bar{r}^1 = \bar{r}_-^1 \oplus \bar{r}_-^1$ and $\bar{\mathcal{G}}^1 = \tilde{\mathcal{G}}/\bar{r}^1$. By the universality, we can show $\bar{\mathcal{G}}^1 = \bar{\mathcal{G}}$. Then we have $\bar{r}_\mp^1 = \bar{r}_\pm$

Q.E.D.

Lemma 2.7.2. *For $\alpha_i \in \Pi$, we have $\dim \bar{\mathcal{G}}_{\alpha_i} = 1$. If $p(\alpha_i) = 1$ and $(\alpha_i,\alpha_i) \neq 0$, then $\dim \mathcal{G}_{2\alpha_i} = 1$.*
Proof. The proof is obtained similarly to the proof of Lemma 1.2.1.

Q.E.D.

Proposition 2.7.2. *Let Φ be the set of the roots of $\mathcal{G}(\mathcal{E},\Pi,p)$ associated with $(\mathcal{E},\Pi,p) \in D(\Theta_N)$. For $\bar{\mathcal{G}} = \bar{\mathcal{G}}(\mathcal{E},\Pi,p)$, let $\bar{\mathcal{G}}_\gamma = \{x \in \bar{\mathcal{G}} \,|\, [H,x] = \gamma(H)x\}$. Then we have*

(i) $\dim \bar{\mathcal{G}}_\gamma = 1$ *if* $\gamma \in \Phi \setminus Z\delta$.
(ii) $\dim \bar{\mathcal{G}}_\gamma \geq \dim \mathcal{G}_\gamma$ *if* $\gamma \in Z\delta$.
(iii) $\dim \bar{\mathcal{G}}_\gamma = 0$ *if* $\gamma \notin \Phi \cup \{0\}$.

In particular,

$$\ker \Psi(\mathcal{E}, \Pi, p) \subset \oplus_{r\neq 0} \bar{\mathcal{G}}_{r\delta}.$$

Proof. By Lemma 2.7.1, we may assume $\gamma \in P_+$. (ii) is clear. Using Lemma 2.7.2, we can proof (i) similarly to the proof of Proposition 2.5.1. Moreover (iii) can be proved similarly to the proof of Proposition 2.5.1: If $\gamma \notin \Phi \cup \{0\}$, then, by $\sigma(i)$'s, we can get a path from γ to $(\oplus_{\alpha_i \in \Pi'} Z\alpha_i) \setminus (P'_+ \cup -P'_+)$ of another datum $(\mathcal{E}', \Pi', p')$. By the triangular decomposition of $\bar{\mathcal{G}}(\mathcal{E}', \Pi', p')$, we have $\dim \bar{\mathcal{G}}_\gamma = 0$.

Q.E.D.

2.8. Proposition 2.8.1. *Let* $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$.

(i) There exists a unique Lie superalgebra $\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p)$ *satisfying following (1), (2) and (3).*

(1) For $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$*, there is a sequence of epimorphisms*

$$\tilde{\mathcal{G}}(\mathcal{E}, \Pi, p) \overset{\tilde{\Psi}^\ddagger(\mathcal{E},\Pi,p)}{\twoheadrightarrow} \bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p) \overset{\Psi^\ddagger(\mathcal{E},\Pi,p)}{\twoheadrightarrow} \mathcal{G}(\mathcal{E}, \Pi, p)$$

$$(H_\gamma, E_i, F_i \ \longrightarrow H_\gamma, E_i, F_i \ \longrightarrow H_\gamma, E_i, F_i).$$

(2) By (1), there is a root space decomposition

$$\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p) = \mathcal{H} \oplus (\oplus_{\alpha \in \Psi^\ddagger} \bar{\mathcal{G}}^\ddagger_\alpha)$$

such that $\mathcal{E} = \mathcal{H}^* \supset \Psi^\ddagger \supset \Psi$*. Here* Ψ *is the set of the roots of* $\mathcal{G}(\mathcal{E}, \Pi, p)$*. Then the assumption (2) is that* $\Psi^\ddagger = \Psi$ *and* $\dim \bar{\mathcal{G}}^\ddagger_\alpha = 1$ *if* $\alpha \in \Psi \setminus Z\delta$.

(3) If there is a Lie superalgebra $\bar{\mathcal{G}}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p)$ *satisfying (1) and (2), then there is an epimorphism:*

$$\bar{\Psi}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p) : \bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p) \twoheadrightarrow \bar{\mathcal{G}}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p)$$

(ii) $\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p)$ *is isomorphic to* $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)$.

Proof. (i) Let $\{\bar{\mathcal{G}}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p)\}_{(\lambda)\in(\Lambda)}$ be the family of Lie superalgebras satisfying (1) and (2). Let $\tilde{\Psi}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p) : \tilde{\mathcal{G}}(\mathcal{E}, \Pi, p) \to \bar{\mathcal{G}}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p)$ be the epimorphism in (1) for $(\lambda) \in (\Lambda)$. Let $\bar{r}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p) = \ker \tilde{\Psi}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p)$ and $\bar{r}^\ddagger(\mathcal{E}, \Pi, p) = \cap_{(\lambda)\in(\Lambda)} \bar{r}^{\ddagger(\lambda)}(\mathcal{E}, \Pi, p)$. Put $\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p) = \tilde{\mathcal{G}}(\mathcal{E}, \Pi, p)/\bar{r}^\ddagger(\mathcal{E}, \Pi, p)$. Then $\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p)$ satisfies (1), (2) and (3).

(ii) By the universality of $\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p)$, there is an epimorphism $L^\ddagger_i$: $\bar{\mathcal{G}}^\ddagger(\mathcal{E}, \Pi, p) \to \bar{\mathcal{G}}^\ddagger(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)})$ such that $L^\ddagger_i(H_\gamma) = H_{\sigma(i)(\gamma)}$, $L^\ddagger_i(E_j) =$

$\Psi^{\ddagger}(\mathcal{E},\Pi,p)^{-1}L_i(E_j)$. $L_i^{\ddagger}(E_j)=\Psi^{\ddagger}(\mathcal{E},\Pi,p)^{-1}L_i(E_j)$. Clearly $\{L_i\}$ satisfy (2) of Proposition 2.7.1. Then we have an epimorphism $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)\to\bar{\mathcal{G}}^{\ddagger}(\mathcal{E},\Pi,p)$. By the universality of $\bar{\mathcal{G}}^{\ddagger}(\mathcal{E},\Pi,p)$, this map is isomorphism.

Q.E.D.

As an immediate consequence, we have:

Lemma 2.8.1. *For $(\mathcal{E},\Pi,p)\in D(\Theta_N)$, there is an epimorphism*

$$\Psi^{\dagger}(\mathcal{E},\Pi,p):\bar{\mathcal{G}}(\mathcal{E},\Pi,p)\twoheadrightarrow\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p).$$

where $\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)$ has already introduced in §1 for $(\mathcal{E},\Pi,p)$ of affine ABCD-type.

In §3, we will show that $\Psi^{\dagger}(\mathcal{E},\Pi,p)$ is isomorphism.

3. The estimation of $\dim\bar{\mathcal{G}}_{r\delta}$

3.1. Proposition 3.1.1. *(i) Let $(\mathcal{E},\Pi,p)\in D(\Theta_N)$. Put $D_N=\sum_{i=1}^N\bar{d}_i$. Then*

$$(\delta,2\rho)=\begin{cases}2D_N & (AA),\\ 2D_N & (BB),\\ 2\bar{d}_1+4D_N & (CB),\\ -2\bar{d}_1+4D_N & (DB),\\ 2\bar{d}_1+4D_N+2\bar{d}_N & (CC),\\ -2\bar{d}_1+4D_N+2\bar{d}_N & (DC),\\ 2\bar{d}_1+4D_N-2\bar{d}_N & (CD),\\ -2\bar{d}_1+4D_N-2\bar{d}_N & (DD).\end{cases}$$

(ii) For exceptional type, we have:

$$(\delta,2\rho)=\begin{cases}0 & D(2,1,;x)^{(1)},\\ -12 & F_4^{(1)},\\ 12 & G_3^{(1)}.\end{cases}$$

Proof. Direct calculations.

Q.E.D.

Here we again remark, if there is a relation of weight $\beta\in P_+$ such that $(\beta,\beta)\neq 2(\beta,\rho)$, then the relation can be obtained by relations of lower weights (see Proposition 1.2.1). Therefore have:

Lemma 3.1.1 If $(\mathcal{E},\Pi,p)\in D(\Theta_N)$ satisfies $(\delta,\delta)\neq 2(\delta,\rho)$, then $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)\cong \mathcal{G}(\mathcal{E},\Pi,p)$.

3.2. Proposition 3.2.1. *Let $(\mathcal{E},\Pi,p)\in D(\Theta_N)[\Gamma]$. Assume that x-th simple root $\alpha_x\in\Pi$ and y-th simple root $\alpha_y\in\Pi$ satisfy $\alpha_x+\alpha_y\in\Phi_+$ (i.e. $(\alpha_x,\alpha_y)\neq 0$), $\alpha_x+\alpha_y=\sigma(x)(\alpha_y^\dagger)$ and that $\Pi_{(x,y)}=(\Pi\setminus\{\alpha_x,\alpha_y\})\cup\{\alpha_x+\alpha_y\}$ is affine $ABCD$ type.*

(i) Let $\Pi^\dagger_{(x,y)}$ be $\Pi^\dagger$ of $\Pi_{(x,y)}$. Then $\Pi^\dagger_{(x,y)}=(\Pi^\dagger\setminus\{\alpha_x^\dagger,\alpha_y^\dagger\})\cup\sigma(x)(\alpha_y^\dagger)$ Let $W_{(x,y)}$ denote a subgroup of W generated by $(\{\sigma(0),\dots,\sigma(n)\}\setminus\{\sigma(x),\sigma(y)\})\cup\{\sigma(x)\sigma(y)\sigma(x)\}$. Then $W_{(x,y)}$ is W defined for $(\mathcal{E},\Pi_{(x,y)},p)$.

(ii) Let $(\mathcal{E},\Pi_{(x,y)},p)$ be a datum such that the set of simple roots is $\Pi_{(x,y)}$. Then there is a homomorphism $i:\bar{\mathcal{G}}(\mathcal{E},\Pi_{(x,y)},p)\to\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$ such that

$$i(H_{\alpha_j})=\begin{cases} H_{\alpha_x+\alpha_y} & \alpha_j=\alpha_x+\alpha_y,\\ H_{\alpha_j} & \alpha_j\neq\alpha_x+\alpha_y,\end{cases}$$

$$i(E_j)=\begin{cases} (-1)^{p(\alpha_x)p(\alpha_y)}(\alpha_x,\alpha_y)^{-1}[E_x,E_y] & \alpha_j=\alpha_x+\alpha_y,\\ E_j & \alpha_j\neq\alpha_x+\alpha_y,\end{cases}$$

$$i(F_j)=\begin{cases} [F_x,F_y] & \alpha_j=\alpha_x+\alpha_y,\\ F_j & \alpha_j\neq\alpha_x+\alpha_y.\end{cases}$$

Proof. (i) We can check the fact for each affine type.

(ii) Let $\{\bar{L}_0,\dots,\bar{L}_n\}$ be the isomorphisms defined in Proposition 2.7.1. We consider an orbit $Orbit(\bar{\mathcal{G}}(\mathcal{E},\Pi,p))_{(x,y)}$ through $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$ under the action of a subgroup generated by $(\{\bar{L}_0,\dots,\bar{L}_n\}\setminus\{\bar{L}_x,\bar{L}_y\})\cup\{\bar{L}_x\bar{L}_y\bar{L}_x\}$. Then $Orbit(\bar{\mathcal{G}}(\mathcal{E},\Pi,p))_{(x,y)}$ satisfies the conditions of $\{\bar{\mathcal{G}}^{(\lambda)}(\mathcal{E},\Pi_{(x,y)},p)\}$ in Proposition 2.7.1 where $\{\bar{L}_i^{(\lambda)}\}$ is $(\{\bar{L}_0,\dots,\bar{L}_n\}\setminus\{\bar{L}_x,\bar{L}_y\})\cup\{\bar{L}_x\bar{L}_y\bar{L}_x\}$. By the universality of $\bar{\mathcal{G}}(\mathcal{E},\Pi_{(x,y)},p)$, we get the homomorphism $i:\bar{\mathcal{G}}(\mathcal{E},\Pi_{(x,y)},p)\to\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$.

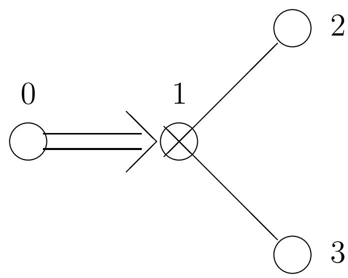
Q.E.D.

3.3. Let $(\mathcal{E},\Pi,p)\in D(\Theta_N)[\Gamma]$. Here we shall show that $\bar{\mathcal{G}}(\mathcal{E},\Pi,p)$ is isomorphic to $\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)$. By Proposition 2.8.1, we have to show $\dim\bar{\bar{\mathcal{G}}}(\mathcal{E},\Pi,p)_{n\delta}=\dim\bar{\mathcal{G}}(\mathcal{E},\Pi,p)_{n\delta}$. *Here we only prove the fact in the case of $(\mathcal{E},\Pi,p)$ of Diagram 1.11.6 with $\sum p(\alpha_i)\equiv 1$.* Because we can prove the fact in another case by the similar way. In this case, $\bar{\bar{\mathcal{G}}}=\widehat{osp}(\tilde{\mathcal{E}}_0,e)^{(I)}$ (see 1.11). Then we have to show:

$$\dim\bar{\mathcal{G}}(\mathcal{E},\Pi,p)_{n\delta}\leq N\quad(n\neq 0).\tag{3.3.1}$$

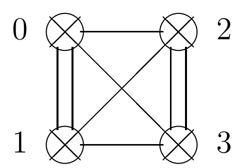
We start with $N=3$. In this case, its Dynkin diagram is:

Diagram 3.3.1



The isomorphism $\sigma(1)$ divert Diagram 3.3.1 into:

Diagram 3.3.2

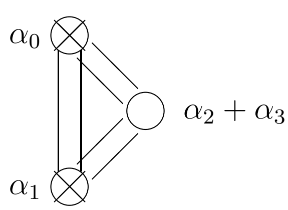


Therefore it is sufficient to show (3.3.1) for $(\mathcal{E}, \Pi, p)$ of Diagram 3.3.2. By Proposition 3.2.1, we have the homomorphism

$$i : \bar{\mathcal{G}}(\mathcal{E}, \Pi_{(2,3)}, p) \to \bar{\mathcal{G}}(\mathcal{E}, \Pi, p).$$

Here the Dynkin diagram of $(\mathcal{E}, \Pi_{(2,3)}, p)$ is:

Diagram 3.3.3



This is equivalent to Diagram 1.6.2 as the Dynkin diagram of the Kac-Moody Lie superalgebra. By Lemma 3.1.1, $\bar{\mathcal{G}}(\mathcal{E}, \Pi_{(2,3)}, p) = \mathcal{G}(\mathcal{E}, \Pi_{(2,3)}, p)$. Hence

$$\dim \bar{\mathcal{G}}(\mathcal{E}, \Pi_{(2,3)}, p)_{n\delta} = 2 \quad (n \neq 0). \tag{3.3.2}$$

We assume $n > 0$. For a root $\gamma \notin Z\delta$ of $\mathcal{G}(\mathcal{E}, \Pi, p)$ (resp. $\mathcal{G}(\mathcal{E}, \Pi_{(2,3)}, p)$), let E_γ (resp. $E_\gamma^{(2,3)}$) denote a non-zero element of $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)_\gamma$ (resp. $\bar{\mathcal{G}}(\mathcal{E}, \Pi_{(2,3)}, p)_\gamma$).

By (3.3.2), $\{[E^{(2,3)}_{n\delta-\alpha_0}, E^{(2,3)}_{\alpha_0}],\ [E^{(2,3)}_{n\delta-\alpha_1}, E^{(2,3)}_{\alpha_1}],\ [E^{(2,3)}_{n\delta-(\alpha_2+\alpha_3)}, E^{(2,3)}_{\alpha_2+\alpha_3}]\}$ are linearly dependent. Transposing by i, we see that $\{[E_{n\delta-\alpha_0}, E_0],\ [E_{n\delta-\alpha_1}, E_1],$ $[E_{n\delta-(\alpha_2+\alpha_3)}, [E_2, E_3]]\}$ are linearly dependent. Since $[E_{n\delta-(\alpha_2+\alpha_3)}, [E_2, E_3]]$ $= [[E_{n\delta-(\alpha_2+\alpha_3)}, E_2], E_3]\ -(-1)^{p(\alpha_2)p(\alpha_3)}[[E_{n\delta-(\alpha_2+\alpha_3)}, E_3], E_2]$, $\{[E_{n\delta-\alpha_0}, E_0],$ $[E_{n\delta-\alpha_1}, E_1], [E_{n\delta-\alpha_2}, E_2], [E_{n\delta-\alpha_3}, E_3]\}$ are linearly dependent. Hence $\dim \bar{\mathcal{G}}(\mathcal{E}, \Pi, p) \leq$ 3. Then we could show (3.3.1) for $N = 3$.

Next we show (3.3.1) for $N \geq 4$ by induction. By Proposition 3.2.1, we can use the homomorphism

$$i : \bar{\mathcal{G}}(\mathcal{E}, \Pi_{(N-3,N-2)}, p) \to \bar{\mathcal{G}}(\mathcal{E}, \Pi, p).$$

Using a similar argument to that in the case of $N = 3$, we can show (3.3.1).

Using a similar argument to that in the above case, we can get:

Theorem 3.3.1. *Let $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$. Then $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)$ is isomorphic to $\bar{\bar{\mathcal{G}}}(\mathcal{E}, \Pi, p)$.*

4. Relations of Affine $ABCD$-types

4.1. Let $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$. Using the definition of $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)$ given in Proposition 2.7.1, we can directly calculate defining relations of $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)$.

Theorem 4.1.1. *Let $(\mathcal{E}, \Pi, p) \in D(\Theta_N)$ (i.e., $(\mathcal{E}, \Pi, p)$ is affine $ABCD$ type). The Lie superalgebra $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p)$ is defined by generators $H \in \mathcal{H}$, E_i, F_i $(0 \leq i \leq n)$ with parities $p(H) = 0$, $p(E_i) = p(E_i) = p(\alpha_i)$ and relations:*

(1) $[H, H'] = 0, \quad (H, H' \in \mathcal{H})$

(2) $[H, E_i] = \alpha_i(H)E_i, \quad [H, F_i] = -\alpha_i(H)F_i,$

(3) $[E_i, F_j] = \delta_{ij}H_{\alpha_i},$

(4) Relations of E_i's.

(i) $[E_i, E_j] = 0$

if $(\alpha_i, \alpha_j) = 0$, $(i \neq j)$,

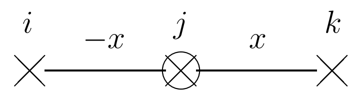
(ii) $[E_i, E_i] = 0$

if $\overset{i}{\bigotimes}$

(iii) $[E_i, [E_i, \ldots, [E_i, E_j] \ldots]] = 0$
(E_i appears $1 - 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ times)

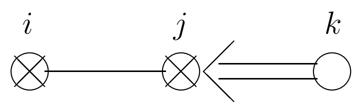
$$\textit{if } (\alpha_i, \alpha_i) \neq 0 \textit{ and } (-1)^{\{p(\alpha_i)\frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}\}} = 1,$$

(iv) $[[[E_i, E_j], E_k], E_j] = 0$

if 

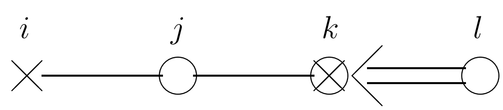
$(x \neq 0)$,

(v) $[[[E_i, E_j], [[E_i, E_j], E_k]], E_j] = 0$

if 

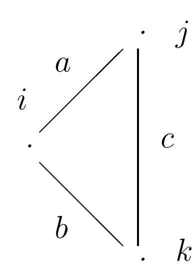
,

(vi) $[[[[[[E_i, E_j], E_k], E_l], E_k], E_j], E_k] = 0$

if 

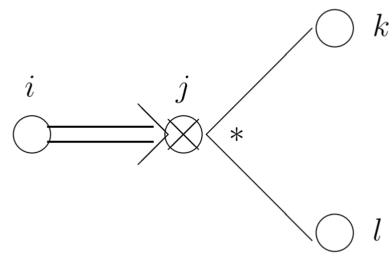
,

(vii) $(-1)^{p(\alpha_i)p(\alpha_k)}(\alpha_i, \alpha_k)[[E_i, E_j], E_k] = (-1)^{p(\alpha_i)p(\alpha_j)}(\alpha_i, \alpha_j)[[E_i, E_k], E_j]$

if 

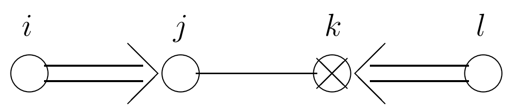
$$(a = (\alpha_i, \alpha_j), b = (\alpha_i, \alpha_k), c = (\alpha_j, \alpha_k),$$
$$abc \neq 0, a + b + c = 0,$$
$$p(\alpha_i)p(\alpha_j) + p(\alpha_i)p(\alpha_j) + p(\alpha_i)p(\alpha_j) \equiv 1),$$

(viii) $[[[E_i,E_j],[E_j,E_k]],[E_j,E_l]]=[[[E_i,E_j],[E_j,E_l]],[E_j,E_k]]$

if 

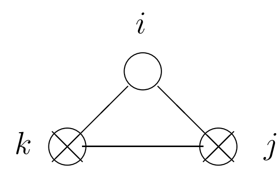
(ix)

$$[[[E_k,[E_l,[E_k,E_j,]]],[E_k,[E_l,[E_k,[E_j,E_i,]]]]],E_j]$$
$$=2[[E_k,E_j],[[E_k,[E_j,E_i]],[E_k,[E_l,[E_k,[E_j,E_i,]]]]]]$$

if  ,

(x)

$$[E_j,[E_k,[E_j,[E_k,E_i]]]]=[E_k,[E_j,[E_k,[E_j,E_i]]]]$$

if  ,

(5) Relations of F_i's defined as the same relations as (4).

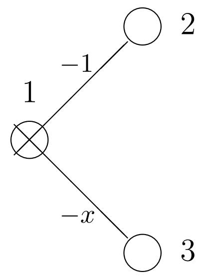
Proof. Direct calculations.

Q.E.D.

5. Relations of $D(2;1,x)^{(1)}$, $F_4^{(1)}$ and $G_3^{(1)}$

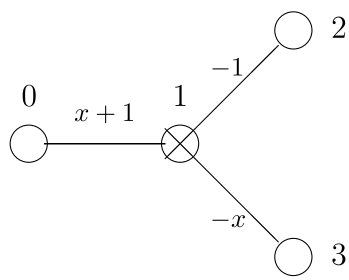
5.1. Let $(\mathcal{E}_0,\Pi_0,p_0)=(\mathcal{E}_0=\oplus_{i=1}^{3}C\alpha_i,\Pi_0=\{\alpha_1,\alpha_2,\alpha_3\},p_0)$ be the datum whose Dynkin diagram is:

Diagram 5.1.1.



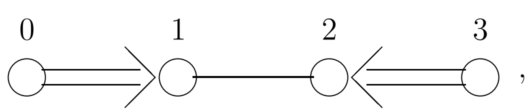
Let $(\mathcal{E}, \Pi, p) = (\mathcal{E}_0, \Pi_0, p_0)^{(I)}$ (see 1.5). Then its Dynkin diagram is:

Diagram 5.1.2.



Using the same argument as the one for affine $ABCD$-type, we can calculate the defining relation of $\mathcal{G}(\mathcal{E}, \Pi, p)$. In the argument, $(\mathcal{E}^\dagger, \Pi^\dagger) = (\mathcal{E}^\dagger = \oplus_{i=0}^3 C\alpha_i^\dagger, \Pi^\dagger = \{\alpha_i^\dagger, (0 \leq i \leq 3)\})$ is defined by a Dynkin diagram:

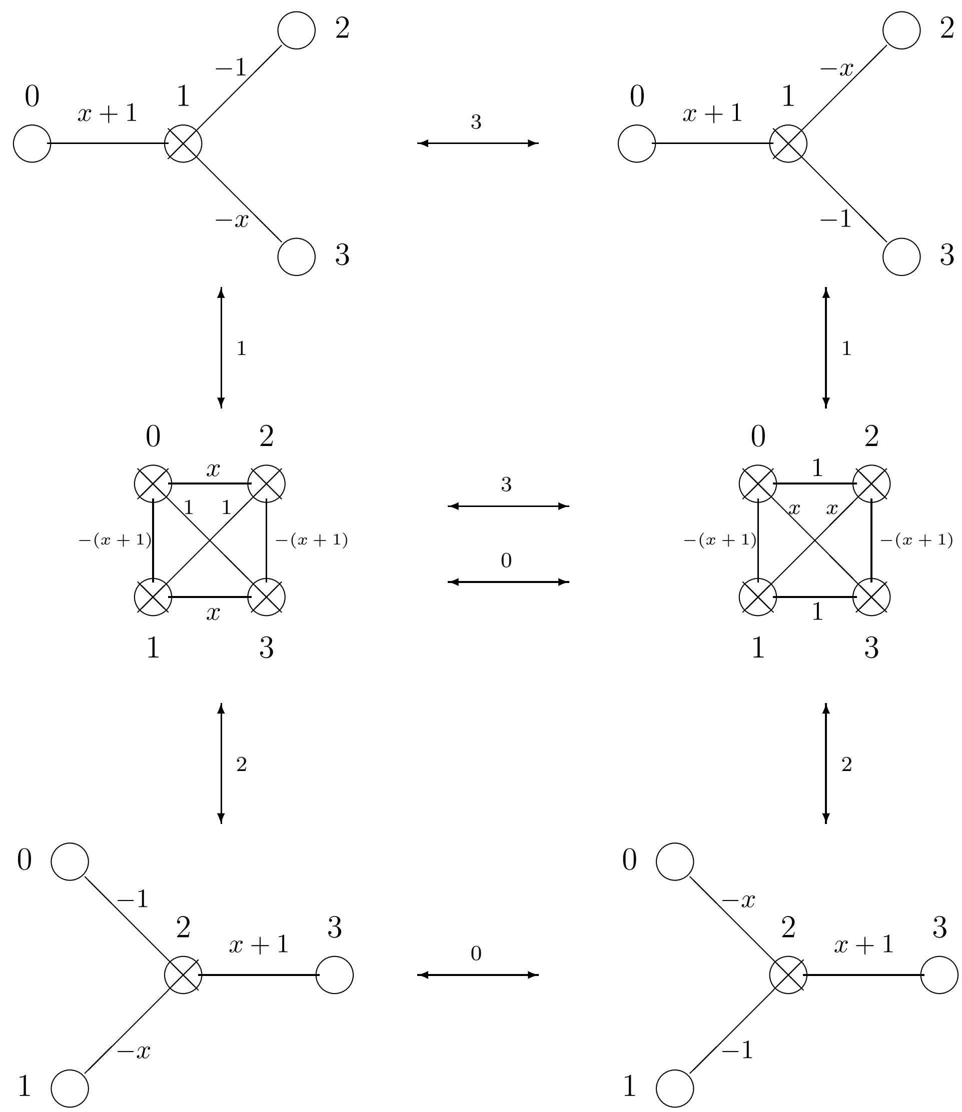
Diagram 5.1.3.



Then the Weyl-group-type isomorphism $\sigma(i)$ and L_i $(0 \leq i \leq 3)$ move as follows;

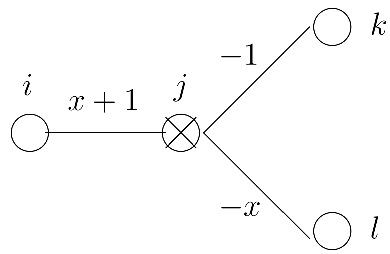
Diagram 5.1.4.

.



Theorem 5.1.1. *Let* $(\mathcal{E}, \Pi, p)$ *be the data of which Dynkin diagrams in Diagram 5.1.4. Then* $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p) = \mathcal{G}(\mathcal{E}, \Pi, p)$ *and its defining relations are ones defined by replacing (vii) of Theorem 4.1.1 with:*

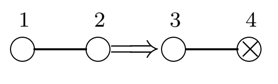
(vii) $[[[E_i, E_j], [E_j, E_k]], [E_j, E_l]] = x[[[E_i, E_j], [E_j, E_l]], [E_j, E_k]]$

if 

$(x \neq \pm 1, 0)$.

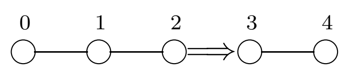
5.2. The same argument can still apply to affine F-type. Let $(\mathcal{E}_0, \Pi_0, p_0)$ $= (\mathcal{E}_0 = \oplus_{i=1}^4 C\alpha_i, \Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, p_0)$ be the datum whose Dynkin diagram is:

Diagram 5.2.1.



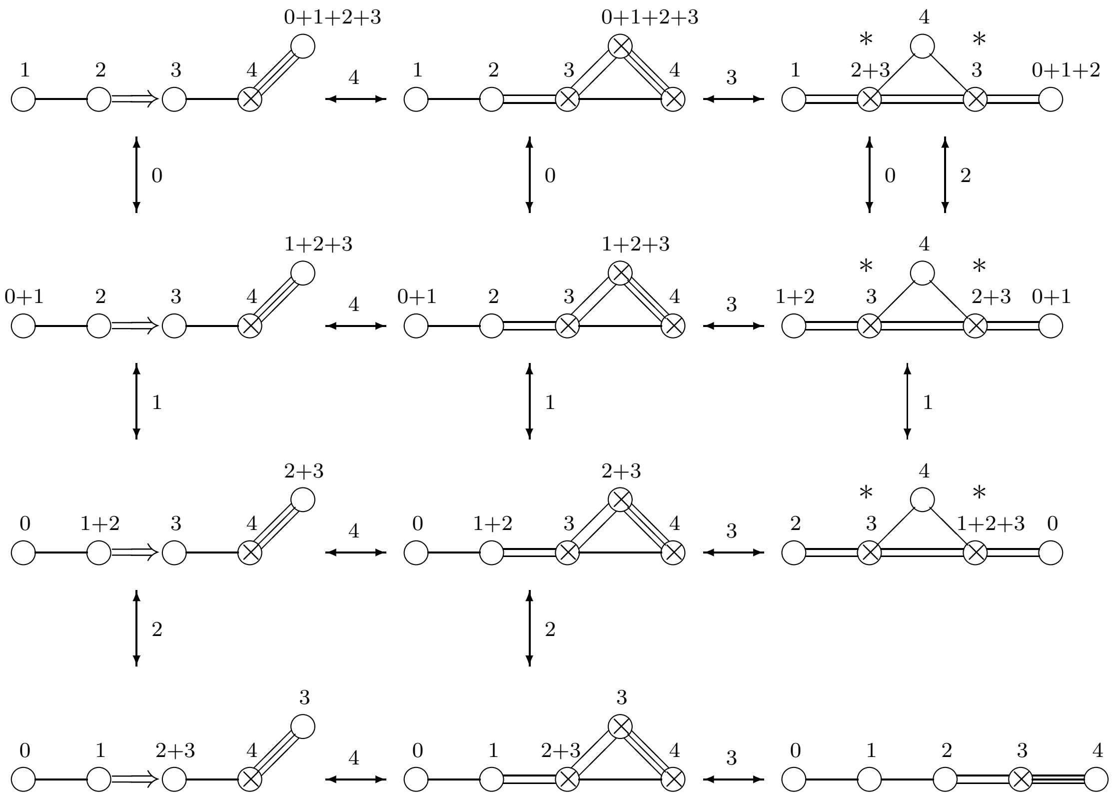
Using the same argument as the one for affine $ABCD$-type, we can calculate the defining relation of $\mathcal{G}(\mathcal{E}, \Pi, p)$. In the argument, $(\mathcal{E}^\dagger, \Pi^\dagger) = (\mathcal{E}^\dagger = \oplus_{i=0}^3, C\alpha_i^\dagger, \Pi^\dagger = \{\alpha_i^\dagger, (0 \leq i \leq 4)\})$ is defined by a Dynkin diagram:

Diagram 5.2.2.



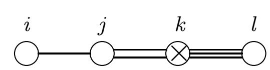
Then the affine F_4 Weyl group type isomorphism $\sigma(i)$ and L_i $(0 \leq i \leq 4)$ move as follows: (In the diagrams, $i + j + \cdots$ denote $\alpha_i^\dagger + \alpha_j^\dagger + \cdots$.)

Diagram 5.2.3.

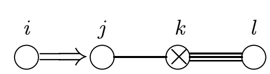


Theorem 5.2.1 *Let $(\mathcal{E}, \Pi, p)$ be the data of which Dynkin diagrams existing in Diagram 5.2.3. Then $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p) = \mathcal{G}(\mathcal{E}, \Pi, p)$ and its relations are defined by adding the following relations to the ones of Theorem 4.1.1 with:*

(4) (xi) $[[[[[[[[[[E_i, E_j], E_k], E_l], E_k], E_j], E_k], E_l], E_k], E_j], E_k] = 0$

if  ,

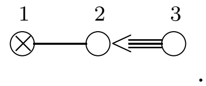
(xii) $[[[[[E_l, E_k], E_j], E_i], E_k], E_j] = 2[[[[[E_l, E_k], E_j], E_i], E_j], E_k]$

if  ,

(5) Relations of F_i's defined as the same relations as (4).

5.3. An argument for affine G-type shall be different from the one for another affine type. Let $(\mathcal{E}_0, \Pi_0, p_0) = (\mathcal{E}_0 = \oplus_{i=1}^{4} C\alpha_i, \Pi_0 = \{\alpha_1, \alpha_2, \alpha_3\}, p_0)$ be the datum whose Dynkin diagram is:

Diagram 5.3.1.

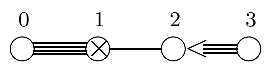


Then the positive roots of $\mathcal{G}(\mathcal{E}_0, \Pi_0, p_0)$ are:

$\Phi_{0,+} = \{a\alpha_1 + b\alpha_2 + c\alpha_3 \,|\, (a,b,c) = (1,0,0),\ (1,1,0),\ (1,1,1),\ (1,2,1),$ $(1,3,1), (1,3,2), (1,4,2), (2,4,2), (0,0,1), (0,1,1), (0,3,2), (0,2,1), (0,3,1),$ $(0,1,0)\}$.

Let $(\mathcal{E}, \Pi, p) = (\mathcal{E}_0, \Pi_0, p_0)^{(I)}$ (see 1.5). Then its Dynkin diagram is:

Diagram 5.3.2.



Here the null root δ of $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$ is given by $\delta = \alpha_0 + 2\alpha_1 + 4\alpha_2 + 2\alpha_3$. For $i = 0, 2, 3$, we define $\sigma(i) : \mathcal{E} \to \mathcal{E}$ by $\sigma(i)(\gamma) = \gamma - \frac{2(\gamma,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. For the set Φ_+ of the positive roots of $\mathcal{G}$, we put

$$\begin{aligned} &\Phi_+^\flat = \Phi_+ \cup \textstyle\bigcup_{i=0}^3 (\Phi_+ + \alpha_i), \\ &\Phi_+^\sharp = \{\gamma \in \Phi_+^\flat | (\gamma,\gamma) = 2(\rho,\gamma)\}. \end{aligned}$$

Then as a more precise fact than (1.2.4), it follows:

$$r_+ = \bigcup_{\gamma \in \Phi_+^\sharp} r_{+,\leq\gamma} \tag{5.3.1}$$

Since $|2(\rho,\delta)| = 12$, it is clear that sufficiently large element of $\Phi_+^\flat$ doesn't belong to $\Phi_+^\sharp$. By direct calculation, we can get:

$\Phi_+^\sharp = \{\alpha_1,\, 2\alpha_1,\, \alpha_0,\, \alpha_3,\, \alpha_2,\, \alpha_0 + 3\alpha_1 + 4\alpha_2 + \alpha_3,\, \alpha_1 + 2\alpha_2,\, \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3,$ $2\alpha_0 + 3\alpha_1 + 5\alpha_2 + 2\alpha_3,\, \alpha_0 + 2\alpha_1 + 4\alpha_2 + 4\alpha_3,\, \alpha_2 + 2\alpha_3,\, \alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3,$ $4\alpha_2 + \alpha_3\}$.

Let $r_+^\sharp$ be the ideal of $\widetilde{\mathcal{N}}^+$ generated by the relations (i), (ii), (iii) of Theorem 4.1.1 and

$$2[[[[[E_0, E_1], E_2], E_3], E_1], E_2] = 3[[[[[E_0, E_1], E_2], E_3], E_2], E_1]\,. \tag{5.3.2}$$

We also define the ideal $r_-^\sharp$ of $\widetilde{\mathcal{N}}^-$ in the same way. By the criterion of Lemma 2.2.2, we can see $r_\pm^\sharp \subset r_\pm$. We can also see that $r^\sharp = r_-^\sharp \oplus r_+^\sharp$ is an

ideal of $\tilde{\mathcal{G}}$. Let $\mathcal{G}^\sharp = \tilde{\mathcal{G}}/r^\sharp$. Then we have a triangular decomposition $\mathcal{G}^\sharp = \mathcal{N}^{\sharp,+} \oplus \mathcal{H} \oplus \mathcal{N}^{\sharp,-}$ where $\mathcal{N}^{\sharp,\pm} = \widetilde{\mathcal{N}}^{\pm}/r^\sharp_{\pm}$. Since the relations (iii) of Theorem 4.1.1 and their F_i's version hold in $\mathcal{G}^\sharp$, the automorphisms $L_i^\sharp : \mathcal{G}^\sharp \to \mathcal{G}^\sharp$ $(i = 0, 2, 3)$ given by

$$L_i^\sharp = \exp(\mathrm{ad}E_i)\exp(\mathrm{ad}(-\frac{2}{(\alpha_i, \alpha_i)}F_i))\exp(\mathrm{ad}E_i)$$

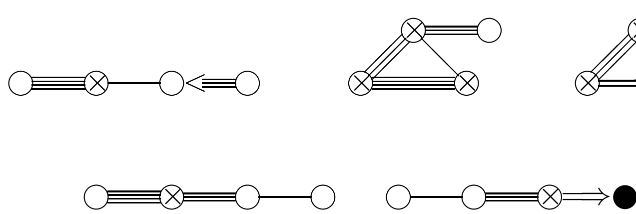
are well-defined (see also [K1]). Let $\mathcal{G}^\sharp = \mathcal{H} \oplus (\oplus_{\alpha \in P_+ \cup P_-} \mathcal{G}^\sharp_\alpha)$ be the root space decomposition. Then we have $L_i^\sharp(\mathcal{G}^\sharp_\alpha) = \mathcal{G}^\sharp_{\sigma(i)(\alpha)}$. By Proposition 2.2.2, we have already had the automorphism $L_i : \mathcal{G} \to \mathcal{G}$ such that $L_i(\mathcal{G}_\alpha) = \mathcal{G}_{\sigma(i)(\alpha)}$. Clearly $\dim \mathcal{G}^\sharp_\alpha = \dim \mathcal{G}_\alpha$ if $\alpha \in \{\alpha_0, \alpha_1, 2\alpha_1, \alpha_2, \alpha_3\}$ Therefore, if $\beta \in P_+$ is a minimal element under the order $\leq$ (see 1.2) among $\dim \mathcal{G}^\sharp_\beta > \dim \mathcal{G}_\beta$, then

$$\beta \in \Phi^\flat_+ \text{ and } (\beta, \alpha_0) \geq 0,\ (\beta, \alpha_2) \leq 0,\ (\beta, \alpha_3) \leq 0 \qquad (5.3.3)$$

because $\sigma(i)(\beta) \leq \beta$ $(i = 0, 2, 3)$. The unique element satisfying (5.3.3) is $\alpha_0+2\alpha_1+2\alpha_2+\alpha_3$. However, using the relation (5.3.2), we see $\dim \mathcal{G}^\sharp_{\alpha_0+2\alpha_1+2\alpha_2+\alpha_3} \leq 1$. Hence $\dim \mathcal{G}^\sharp_{\alpha_0+2\alpha_1+2\alpha_2+\alpha_3} = \dim \mathcal{G}_{\alpha_0+2\alpha_1+2\alpha_2+\alpha_3}$. Hence such β doesn't exist. Hence $\mathcal{G}^\sharp$ is isomorphic to $\mathcal{G}$.

By Proposition 2.2.1,we can get other Dynkin diagrams of $\mathcal{G}$. Those are:

Diagram 5.3.3.



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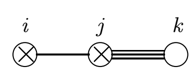
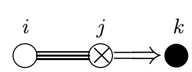
Theorem 5.3.1. *Let $(\mathcal{E}, \Pi, p)$ be the data of which Dynkin diagrams existing in Diagram 5.2.3. Then defining relations of $\mathcal{G}(\mathcal{E}, \Pi, p)$ are defined by adding the following relations to the ones of Theorem 4.1.1 with:*

(4) (xiii) $[[[E_i, E_j], [[E_i, E_j], [[E_i, E_j], E_k]]], E_j] = 0$

if 

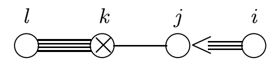
,

(xix) $[E_j, [E_k, [E_k, [E_j, E_i]]]] = [E_k, [E_j, [E_k, [E_j, E_i]]]]$

if 

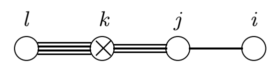
,

(xx) $2[[[[[E_l, E_k], E_j], E_i], E_k], E_j] = 3[[[[[E_l, E_k], E_j], E_i], E_j], E_k]$

if 

,

(xxi)
$[[[[[[[[[[[[[[E_i, E_j], E_k], E_l], E_k], E_j], E_k], E_l], E_k], E_j], E_k], E_l], E_k], E_j], E_k] = 0$

if 

,

(5) Relations of F_i's defined as the same relations as (4).

6. Quantization of relations

6.1. Let $(\mathcal{E}, \Pi, p)$ be a datum. Let $C[[h]]$ denote the C-algebra of formal power series in h. In [Y1], we defined an h-adic topological Hopf superalgebra $U_h(\mathcal{G}) = U_h(\mathcal{G}(\mathcal{E}, \Pi, p))$ in an abstract manner. For the terminologies of topological algebras, Hopf superalgebras etc., see [Y1]. Let $\tilde{U}_h^\flat(\mathcal{B}^+)^\sigma$ be a non-topological $C[[h]]$-algebra defined with generators K_λ ($\lambda \in Z\Pi$), E_α ($\alpha \in \Pi$), σ and relations:

$$\sigma^2 = 1,\ \sigma K_\lambda \sigma = K_\lambda,\ \sigma E_\alpha \sigma = (-1)^{p(\alpha)} E_\alpha,$$
$$K_0 = 1,\ K_\lambda K_\mu = K_{\lambda+\mu},\ K_\lambda E_\alpha {K_\lambda}^{-1} = \exp((\lambda, \alpha)h) E_\alpha.$$

It is easy to see that $\tilde{U}_h^\flat(\mathcal{B}^+)^\sigma$ is a Hopf algebra with coproduct Δ, antipode S and counit ε such that

$$\Delta(\sigma) = \sigma \otimes \sigma,\ \Delta(K_\lambda) = K_\lambda \otimes K_\lambda,\ \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \sigma^{p(\alpha)} \otimes E_\alpha$$
$$S(\sigma) = \sigma,\ S(K_\lambda) = K_\lambda^{-1},\ S(E_\alpha) = -K_\alpha^{-1} \sigma^{p(\alpha)} E_\alpha$$
$$\varepsilon(\sigma) = 1,\ \varepsilon(K_\lambda) = 1,\ \varepsilon(E_\alpha) = 0.$$

We note that $\tilde{U}_h^\flat(\mathcal{B}^+)^\sigma$ is not a Hopf superalgebra but a Hopf algebra.

By [Y1], we have:

Lemma 6.1.1. *(i)*

$$E_{\alpha(1)} \cdots E_{\alpha(r)} K_\lambda \sigma^c \qquad (\alpha(j) \in \Pi,\ \lambda \in Z\Pi,\ c \in \{0, 1\})$$

form a $C[[h]]$-basis of $\tilde{U}_h^\flat(\mathcal{B}^+)^\sigma$. In particular, as topological modules,

$$\tilde{U}_h^\flat(\mathcal{B}^+)^\sigma \cong \tilde{N}^+ \otimes C[[h]]\,[K_\lambda] \otimes C[[h]]\langle\sigma\rangle. \tag{6.1.1}$$

Here $\tilde{N}^+$, $C[[h]]\,[K_\lambda]$ and $C[[h]]\langle\sigma\rangle$ denote the free algebra generated by E_α ($\alpha \in \Pi$), the Laurent polynomial algebra in $K_\alpha^{\pm 1}$ ($\alpha \in \Pi$) and the group ring of $\{1, \sigma\}$ respectively.

(ii) There is a symmetric Hopf pairing $\langle\,,\,\rangle : \tilde{U}_h^\flat(\mathcal{B}^+)^\sigma \times \tilde{U}_h^\flat(\mathcal{B}^+)^\sigma \to C[[h]]$ *such that;*
(a) $\tilde{N}^+$ *and* $C[[h]]\,[K_\lambda] \otimes C[[h]]\langle\sigma\rangle$ *are orthogonal,*
(b) $\langle E_\alpha, E_\beta\rangle = \delta_{\alpha,\beta}$ $(\alpha, \beta \in \Pi)$,
(c) $\langle K_\lambda\sigma^c, K_\mu\sigma^d\rangle = \exp((\lambda,\mu)h)(-1)^{cd}$.

Remark 6.1.1 In [Y1], we introduced an another topological Hopf algebra $\tilde{U}'_{\sqrt{h}}(b_+)^\sigma$. Then $\tilde{U}_h^\flat(\mathcal{B}^+)^\sigma$ is given as the non-topological subalgebra of $\tilde{U}'_{\sqrt{h}}(b_+)^\sigma$ generated by E_α, $K_\alpha^\pm = \exp(\pm\sqrt{h}H'_\alpha)$ and σ.

Lemma 6.1.2. *(See [Y1]) Let* $I^+ = \{X \in \tilde{N}^+ | \langle X, Y\rangle = 0\,(Y \in \tilde{N}^+)\}$*. Then* $\ker\langle\,,\,\rangle \cong I^+ \otimes C[[h]]\,[K_\lambda] \otimes C[[h]]\langle\sigma\rangle$ *under (6.1.1). In particular, letting* $U_h^\flat(\mathcal{B}^+)^\sigma = \tilde{U}_h^\flat(\mathcal{B}^+)^\sigma / \ker\langle\,,\,\rangle$ *and* $N^+ = \tilde{N}^+/I^+$*, it follows that* $U_h^\flat(\mathcal{B}^+)^\sigma \cong N^+ \otimes C[[h]]\,[K_\lambda] \otimes C[[h]]\langle\sigma\rangle$.

By [Y1] and Lemma 6.1.1, we have:

Theorem 6.1.1. *For the datum* $(\mathcal{E}, \Pi, p)$*, there is an h-adic topological* $C[[h]]$*-Hopf superalgebra* $U_h(\mathcal{G}) = U_h(\mathcal{G}(\mathcal{E}, \Pi, p))$ *with generators* H_λ $(\lambda \in \mathcal{E})$, E_α, F_α $(\alpha \in \Pi)$ *with parities* $p(H_\lambda) = 0$, $p(E_\alpha) = p(F_\alpha) = p(\alpha)$ *satisfying following (a) and (b):*

(a) In $U_h(\mathcal{G})$*, we have:*

$$\begin{aligned}&[H_\lambda, H_\mu] = 0,\ [H_\lambda, E_\alpha] = (\lambda,\alpha)E_\alpha,\ [H_\lambda, F_\alpha] = -(\lambda,\alpha)F_\alpha,\\ &[E_\alpha, F_\beta] = \delta_{\alpha,\beta}\tfrac{\sinh(hH_\alpha)}{\sinh(h)}.\end{aligned} \tag{6.1.3}$$

Put $K_\lambda = \exp(hH_\lambda)$*. Then* $(U_h(\mathcal{G}), \Delta, S, \varepsilon)$ *is a Hopf superalgebra such that*

$$\begin{aligned}&\Delta(H_\lambda) = H_\lambda \otimes 1 + 1 \otimes H_\lambda,\ \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha,\\ &\qquad \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha,\\ &\qquad S(H_\lambda) = -H_\lambda,\ S(E_\alpha) = -K_\alpha^{-1}E_\alpha,\ S(F_\alpha) = -F_\alpha K_\alpha,\\ &\qquad \varepsilon(H_\lambda) = \varepsilon(E_\alpha) = \varepsilon(F_\alpha) = 0.\end{aligned} \tag{6.1.4}$$

(b) Let $C[[h]]\,[\mathcal{H}]$ *(resp.* N^+ *or* N^-*) be the (non-topological) subalgebra of* $U_h(\mathcal{G})$ *generated by* H_λ *(resp.* E_α *or* F_α*). Then* $C[[h]]\,[\mathcal{H}]$ *is the polynomial ring in* $H_\lambda \in \mathcal{H} = \mathcal{E}^*$*. There are algebra isomorphisms* $\tilde{N}^+/I^+ \to N^+$ $(E_\alpha \to E_\alpha)$ *and* $N^+ \to N^-$ $(E_\alpha \to F_\alpha)$*. There is a topological module isomorphism;* $U_h(\mathcal{G}) \leftarrow (N^- \otimes C[[h]]\,[\mathcal{H}] \otimes N^+)^\wedge$ $\quad (YQX \leftarrow Y \otimes Q \otimes X)$*. (Here* $(\,)^\wedge$ *denotes completion.) In particular,* $U_h(\mathcal{G})$ *is topologically free as an h-adic module.*

6.2. Let $U_h(\mathcal{G}) = \oplus_{\gamma \in Z\Pi} U_h(\mathcal{G})_\gamma$ be the weight space decomposition. (Here

$U_h(\mathcal{G})_\gamma = \{X \in U_h(\mathcal{G}) | [H_\lambda, X] = (\lambda,\gamma)X \, (\lambda \in \mathcal{E})\}$. Putting $N^+_\gamma = N^+ \cap U_h(\mathcal{G})_\gamma$, we have $N^+ = \oplus_{\gamma\in P_+} N^+_\gamma$.

Lemma 6.2.1. *There is an anti-homomorphism $t : N^+ \to N^+$ $(E_\alpha \to E_\alpha)$ $(\alpha \in \Pi)$.*

Proof. Let S be the antipode of $U_h^\flat(\mathcal{B}^+)^\sigma$. Define t by putting $t(X) = (-1)^{\sum_{i<j} p(\alpha(i))p(\alpha(j))} \cdot \exp(-\sum_{i<j}(\alpha(i),\alpha(j))h) K_\gamma S(X)$ for $X \in \mathcal{N}^\gamma$ with $\gamma = \sum \alpha(i)$ $(\alpha(i) \in \Pi)$. By Lemma 6.1.2, we can easily check that t is the anti-homomorphism because S is so.

Q.E.D.

Let $C((h))$ be the quotient field of $C[[h]]$. Put $C((h))\,[\mathcal{H}] = (C[[h]]\,[\mathcal{H}])^\wedge \otimes_{C[[h]]} C((h))$. Let $C((h))\,[K_\lambda]$ be the Laurent polynomial $C((h))$-algebra in $K_\alpha^{\pm 1}$ $(\alpha \in \Pi)$. Then $C((h))\,[K_\lambda] = C[[h]]\,[K_\lambda] \otimes_{C[[h]]} C((h))$ and there is an epimorphism $C((h))\,[K_\lambda] \hookrightarrow C((h))\,[\mathcal{H}]$ $(K_\lambda \to \exp(hH_\lambda))$ $(\lambda \in Z\Pi)$. We define $e : U_h(\mathcal{G}) \to C((h))\,[\mathcal{H}]$ as the composition:

$$U_h(\mathcal{G}) \cong (N^- \otimes C[[h]]\,[\mathcal{H}] \otimes N^+)^\wedge \overset{\varepsilon\otimes id\otimes\varepsilon}{\longrightarrow} C[[h]]\,[\mathcal{H}] \hookrightarrow C((h))\,[\mathcal{H}]\,.$$

For $\gamma \in Z\Pi$ and $T \in C((h))\,[K_\lambda]$, denote the coefficient of K_γ of T by $Q_\gamma(T) \in C((h))$. denote the isomorphism $N^- \to N^+$ $(F_\alpha \to E_\alpha)$ by j. Put $q = \exp(h)$ and $q^a = \exp(ah)$. Let $\langle,\rangle : N^+ \times N^+ \to C[[h]]$ be the non-degenerate symmetric pairing induced from $\langle,\rangle$ of Lemma 6.1.1 (ii).

Lemma 6.2.2. *Let $\gamma = \sum_{\alpha\in\Pi} l_\alpha \alpha \in P_+$ $(l_\alpha \in Z_+)$. Let $X \in N^+_\gamma$ and $Y \in N^-_\gamma$. Then $e(XY) \in C((h))\,[K_\lambda]$ and we have:*

$$Q_\gamma(e(XY)) = \frac{\langle t(X), j(Y)\rangle}{(q-q^{-1})^{\sum l_\alpha}} \in C((h)). \tag{6.2.1}$$

Proof. The first statement is clear. Describe $\gamma \in P_+$ as two sums $\gamma = \sum_{i=1}^r \alpha(i) = \sum_{i=1}^r \alpha'(i)$ $(\alpha(i), \alpha'(i) \in \Pi)$. We compare two calculations of $\langle E_{\alpha(r)}E_{\alpha(r-1)}\cdots E_{\alpha(1)}, E_{\alpha'(1)}\cdots E_{\alpha'(r-1)}E_{\alpha'(r)}\rangle$ and $E_{\alpha(r)}\cdots E_{\alpha(r-1)}E_{\alpha(r)}F_{\alpha'(1)}\cdots F_{\alpha'(r-1)}F_{\alpha'(r)}$. By $\Delta(E_{\alpha(i)}) = E_{\alpha(i)} \otimes 1 + K_{\alpha(i)}\sigma^{p(\alpha(i))} \otimes E_{\alpha(i)}$, we can calculate:

$$\begin{aligned}
&\langle E_{\alpha(r)}E_{\alpha(r-1)}\cdots E_{\alpha(1)}, E_{\alpha'(1)}\cdots E_{\alpha'(r-1)}E_{\alpha'(r)}\rangle \\
&= \quad \langle E_{\alpha(r)} \otimes E_{\alpha(r-1)}\cdots E_{\alpha(1)}, \Delta(E_{\alpha'(1)}\cdots E_{\alpha'(r-1)}E_{\alpha'(r)})\rangle \\
&= \sum_{\substack{1\le x\le r\\ \alpha(r)=\alpha'(x)}} \langle E_{\alpha(r)} \otimes E_{\alpha(r-1)}\cdots E_{\alpha(1)}, \\
&\quad K_{\sum_{i=1}^{x-1}\alpha'(i)}\sigma^{p(\sum_{i=1}^{x-1}\alpha'(i))}E_{\alpha'(x)}K_{\sum_{i=x+1}^{r}\alpha'(i)}\sigma^{p(\sum_{i=x+1}^{r}\alpha'(i))} \otimes E_{\alpha'(1)} \overset{\overset{x}{\vee}}{\cdots} E_{\alpha'(r)}\rangle \\
&= \sum_{\substack{1\le x\le r\\ \alpha(r)=\alpha'(x)}} (-1)^{p(\alpha(r))p(\sum_{i=x+1}^{r}\alpha'(i))} q^{-(\alpha(r),\sum_{i=x+1}^{r}\alpha'(i))} \langle E_{\alpha(r-1)}\cdots E_{\alpha(1)}, E_{\alpha'(1)} \overset{\overset{x}{\vee}}{\cdots} E_{\alpha'(r)}\rangle
\end{aligned}$$

and

$$\begin{aligned}
&E_{\alpha(1)}\cdots E_{\alpha(r-1)}E_{\alpha(r)}F_{\alpha'(1)}\cdots F_{\alpha'(r)}\\
&=(-1)^{p(\alpha(r))p(\sum_{i=1}^{r}\alpha'(i))}E_{\alpha(1)}\cdots E_{\alpha(r-1)}F_{\alpha'(1)}\cdots F_{\alpha'(r)}E_{\alpha(r)}\\
&+\frac{1}{(q-q^{-1})}\sum_{\substack{1\leq x\leq r\\ \alpha(r)=\alpha'(x)}}(-1)^{p(\alpha(r))p(\sum_{i=1}^{x-1}\alpha'(i))}\\
&\Big\{q^{-(\alpha(r),\sum_{i=x+1}^{r}\alpha'(i))}E_{\alpha(1)}\cdots E_{\alpha(r-1)}F_{\alpha'(1)}\overset{\overset{x}{\vee}}{\cdots}F_{\alpha'(r)}K_{\alpha(r)}\\
&-q^{(\alpha(r),\sum_{i=x+1}^{r}\alpha'(i))}E_{\alpha(1)}\cdots E_{\alpha(r-1)}F_{\alpha'(1)}\overset{\overset{x}{\vee}}{\cdots}F_{\alpha'(r)}K_{\alpha(r)}^{-1}\Big\}.
\end{aligned}$$

Since $(-1)^{p(\alpha(r))p(\sum_{i=1}^{x-1}\alpha'(i))}=(-1)^{p(\alpha(r))+p(\gamma)p(\alpha(r))}(-1)^{p(\sum_{x+1}^{r}\alpha'(i))}$, we can reach (6.2.1) inductively.

Q.E.D.

Proposition 6.2.1. *If $X\in N^+$ satisfies*

$$[X,F_\alpha]=0\quad \textit{for all } \alpha\in\Pi, \tag{6.2.2}$$

then $X=0$.

Proof. Since $[N^+_\gamma,F_\alpha]\subset N^+_{\gamma-\alpha}$, we may assume $X\in N^+_\gamma$ for some $\gamma\in P_+$. By (6.2.2), we see that $[X,F_{\alpha(1)}F_{\alpha(2)}\cdots F_{\alpha(r)}]=0$ for any $\{\alpha(i)\}$ with $\sum_{i=1}^{r}\alpha(i)=\gamma$. Hence, by Lemma 6.2.2, it follow that $\langle X,X_1\rangle=0$ for any $X_1\in N^+_\gamma$. It is clear that the decomposition $N^+=\oplus_{\gamma\in P_+}N^+_\gamma$ is orthogonal. Hence $X\in\ker\langle\,,\rangle$ whence $X=0$.

Q.E.D.

6.3. For $X_\alpha\in U_h(\mathcal{G})_\alpha$, $X_\beta\in U_h(\mathcal{G})_\beta$, we put:

$$[\![X_\alpha,X_\beta]\!]=X_\alpha X_\beta-(-1)^{p(\alpha)p(\beta)}q^{-(\alpha,\beta)}X_\beta X_\alpha.$$

Let $[x]=\frac{\sinh(xh)}{\sinh(h)}\in C[[h]]$. By Proposition 6.2.1 and direct calculation, we have:

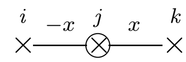
Proposition 6.3.1. *In N^+, we have:*

(i) $[E_i,E_j]=0$ *if* $(\alpha_i,\alpha_j)=0,(i\neq j)$

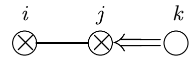
(ii) $[E_i,E_i]=0$ *if* $\overset{i}{\otimes}$,

(iii) $[\![E_i,[\![E_i,\ldots,[\![E_i,E_j]\!]\ldots]\!]]\!]=0$ *(E_i appear $1-\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ times)*

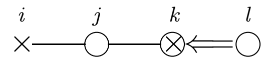
$$if(\alpha_i,\alpha_i)\neq 0 \; and \, p(\alpha_i)\tfrac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} \; is\ even,$$

(iv) $[[\![[\![E_i, E_j]\!], E_k]\!], E_j] = 0$ *if* 

$(x \neq 0)$,

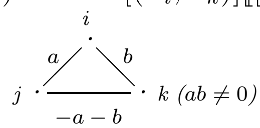
(v) $[[\![E_i, E_j]\!], [\![[\![E_i, E_j]\!], E_k]\!]], E_j] = 0$ *if* 

,

(vi) $[[\![[\![[\![[\![E_i, E_j]\!], E_k]\!], E_l]\!], E_k]\!], E_j]\!], E_k] = 0$ *if* 

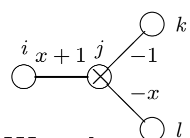
,

(vii) $(-1)^{p(\alpha_i)p(\alpha_k)}[(\alpha_i,\alpha_k)][\![[\![E_i, E_j]\!], E_k]\!] = (-1)^{p(\alpha_i)p(\alpha_j)}[(\alpha_i,\alpha_j)][\![[\![E_i, E_k]\!], E_j]\!]$

if 

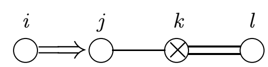
and $p(\alpha_i)p(\alpha_j) + p(\alpha_i)p(\alpha_j) + p(\alpha_i)p(\alpha_j) \equiv 1$,

(viii) $[\![[\![[\![E_i, E_j]\!], [\![E_j, E_k]\!]]\!], [\![E_j, E_l]\!]]\!] = [x][\![[\![[\![E_i, E_j]\!], [\![E_j, E_l]\!]]\!], [\![E_j, E_k]\!]]\!]$

if 

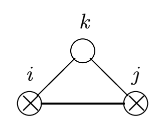
$(x \neq \pm 1,\, 0)$,

(ix) $[\![[\![E_k, [\![E_l, [\![E_k, E_j,]\!]]\!]]\!], [\![E_k, [\![E_l, [\![E_k, [\![E_j, E_i,]\!]]\!]]\!]]\!]]\!], E_j]$
$= [2][\![[\![E_k, E_j]\!], [\![[\![E_k, [\![E_j, E_i]\!]]\!], [\![E_k, [\![E_l, [\![E_k, [\![E_j, E_i,]\!]]\!]]\!]]\!]]\!]]\!]$

if 

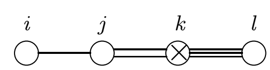
,

(x) $[\![E_j, [\![E_k, [\![E_j, [\![E_k, E_i]\!]]\!]]\!]]\!] = [\![E_k, [\![E_j, [\![E_k, [\![E_j, E_i]\!]]\!]]\!]]\!]$

if 

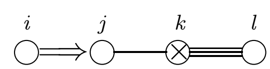
,

(xi) $[[\![[\![[\![[\![[\![[\![[\![[\![E_i, E_j]\!], E_k]\!], E_l]\!], E_k]\!], E_j]\!], E_k]\!], E_l]\!], E_k]\!], E_j]\!], E_k] = 0$

if 

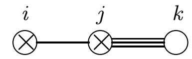
,

(xii) $[\![[\![[\![[\![[\![E_l, E_k]\!], E_j]\!], E_i]\!], E_k]\!], E_j]\!] = [2][\![[\![[\![[\![[\![E_l, E_k]\!], E_j]\!], E_i]\!], E_j]\!], E_k]\!]$

if 

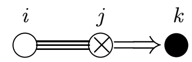
,

(xiii) $[[\![[\![E_i, E_j]\!], [\![[\![E_i, E_j]\!], [\![[\![E_i, E_j]\!], E_k]\!]]\!]]\!], E_j] = 0$

if 

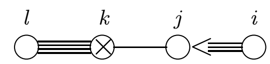
,

(xix) $\llbracket E_j, \llbracket E_k, \llbracket E_k, \llbracket E_j, E_i \rrbracket\rrbracket\rrbracket\rrbracket = \llbracket E_k, \llbracket E_j, \llbracket E_k, \llbracket E_j, E_i \rrbracket\rrbracket\rrbracket\rrbracket$

if 

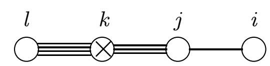
,

(xx) $[2]\llbracket\llbracket\llbracket\llbracket\llbracket E_l, E_k\rrbracket, E_j\rrbracket, E_i\rrbracket, E_k\rrbracket, E_j\rrbracket = [3]\llbracket\llbracket\llbracket\llbracket\llbracket E_l, E_k\rrbracket, E_j\rrbracket, E_i\rrbracket, E_j\rrbracket, E_k\rrbracket$

if 

,

(xxi)
$[\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket\llbracket E_i, E_j\rrbracket, E_k\rrbracket, E_l\rrbracket, E_k\rrbracket, E_j\rrbracket, E_k\rrbracket, E_l\rrbracket, E_k\rrbracket, E_j\rrbracket, E_k\rrbracket, E_l\rrbracket, E_k\rrbracket, E_j\rrbracket, E_k] = 0$

if 

.

In 6.4, we shall describe how we calculate the relations (i)-(xix).

6.4. Let $\mathcal{S}$ be a $C[[h]]$ or $C((h))$-superalgebra. For $a \in C[[h]]^\times$, we put:

$$[X, Y]_a = XY - (-1)^{p(X)p(Y)} aYX \quad (X,\, Y \in \mathcal{S}).$$

Then we have

$$[[X, Y]_a, Z]_b = [X, [Y, Z]_c]_{abc^{-1}} + (-1)^{p(Y)p(Z)} c[[X, Z]_{bc^{-1}}, Y]_{ac^{-1}}, \qquad (6.4.1)$$

and

$$[X, [Y, Z]_a]_b = [[X, Y]_c, Z]_{abc^{-1}} + (-1)^{p(X)p(Y)} c[Y, [X, Z]_{bc^{-1}}]_{ac^{-1}}\,. \qquad (6.4.2)$$

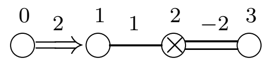
Hence, for $U_h(\mathcal{G})$ and $X_\nu \in U_h(\mathcal{G})_\nu$, $X_\mu \in U_h(\mathcal{G})_\mu$, $X_\eta \in U_h(\mathcal{G})_\eta$, we have:

$$\llbracket\llbracket X_\nu, X_\mu\rrbracket, X_\eta\rrbracket = \llbracket X_\nu, \llbracket X_\mu, X_\eta\rrbracket\rrbracket + (-1)^{p(\mu)p(\eta)} q^{-(\mu,\eta)} [\llbracket X_\nu, X_\eta\rrbracket, X_\mu]_{q^{(\mu,\eta-\nu)}}, \qquad (6.4.3)$$

and

$$\llbracket X_\nu, \llbracket X_\mu, X_\eta\rrbracket\rrbracket = \llbracket\llbracket X_\nu, X_\mu\rrbracket, X_\eta\rrbracket + (-1)^{p(\mu)p(\nu)} q^{-(\mu,\nu)} [X_\mu, \llbracket X_\nu, X_\eta\rrbracket]_{q^{(\mu,\nu-\eta)}}. \qquad (6.4.4)$$

We can get the relations (i)-(xix) of Proposition 6.3.1 by Proposition 6.2.1 and direct calculation using (6.4.3-4). Here we only show how to get (ix) because the other relations can be also gotten similarly. We replace the letters i, j, k, l with 0, 1, 2, 3. We assume $(\alpha_1, \alpha_1) = -2$. Then the diagram can be rewritten as:

 .

Put $E_{\dots dcba} = \llbracket \dots \llbracket E_d, \llbracket E_c, \llbracket E_b, E_a \rrbracket \rrbracket \rrbracket_{\dots} \rrbracket$. Then (ix) is rewritten as:

$$-\llbracket \llbracket E_{2321}, E_{23210} \rrbracket, E_1 \rrbracket + (q+q^{-1}) \llbracket E_{21} \llbracket E_{321}, E_{23210} \rrbracket \rrbracket = 0 \tag{6.4.5}$$

We denote the LHS of (6.4.5) by $\mathcal{X}$. Showing (6.2.2) is equivalent to showing $\llbracket \mathcal{X}, F_a K_a^{-1} \rrbracket = 0$ for all $0 \le a \le 3$ where we put $K_a = K_{\alpha_a}$. We note:

$$\llbracket E_\alpha, F_\beta K_\beta^{-1} \rrbracket = \delta_{\alpha,\beta} \frac{1-K_\beta^{-2}}{q-q^{-1}}. \tag{6.4.6}$$

First we show $\llbracket \mathcal{X}, F_3 K_3^{-1} \rrbracket = 0$. In following equations, $LHS \sim RHS$ mean $LHS = a \cdot RHS$ for some $a \in C[[h]]^\times$. By (6.4.3) and (6.4.6), $\llbracket E_{321}, F_3 K_3^{-1} \rrbracket$ $\sim [\frac{1-K_3^{-2}}{q-q^{-1}}, E_{21}]_{q^{(2+2)}} \sim E_{21}$. Hence

$$\begin{aligned} &\llbracket E_{2321}, F_3 K_3^{-1} \rrbracket \\ &\quad \sim \llbracket E_2, \llbracket E_{321}, F_3 K_3^{-1} \rrbracket \rrbracket \quad \text{by (6.4.3)} \\ &\quad \sim \llbracket E_2, E_{21} \rrbracket \\ &\quad = 0 \quad \text{by (ii) of Proposition 6.3.1.} \end{aligned}$$

Hence we also have $\llbracket E_{23210}, F_3 K_3 \rrbracket = 0$. Hence we have $\llbracket \llbracket \llbracket E_{2321}, E_{23210} \rrbracket, E_1 \rrbracket, F_3 K_3^{-1} \rrbracket = 0$. On the other hand,

$$\begin{aligned} &\llbracket \llbracket E_{21} \llbracket E_{321}, E_{2321} \rrbracket \rrbracket, F_3 K_3^{-1} \rrbracket \\ &\quad \sim \llbracket E_{21} \llbracket E_{21}, E_{23210} \rrbracket \rrbracket \quad \text{by (6.4.3) and } E_{321}^2 = 0 \\ &\quad = 0 \quad \text{since } E_{21}^2 = 0. \end{aligned}$$

Then we have $\llbracket \mathcal{X}, F_3 K_3 \rrbracket = 0$.

Next we show $\llbracket \mathcal{X}, F_2 K_2 \rrbracket = 0$. First we calculate:
$\llbracket E_{2321}, F_2 K_2^{-1} \rrbracket = \llbracket \llbracket E_2, E_{321} \rrbracket, F_2 K_2^{-1} \rrbracket = -q^{-1} [\frac{1-K_2^{-2}}{q-q^{-1}}, E_{321}]_{q^{(1+1)}} = E_{321}$,

$\llbracket E_{23210}, F_2 K_2^{-1} \rrbracket = E_{3210}$,

$$\begin{aligned} &\llbracket \llbracket E_{2321}, E_{23210} \rrbracket, F_2 K_2^{-1} \rrbracket = \llbracket E_{2321}, E_{3210} \rrbracket + q^{-1} [E_{321}, E_{23210}]_{q^{(1-0)}} \\ &\quad = -q^{-2} [E_{23210}, E_{321}]_{q^{(2+1)}} + q^{-1} [E_{321}, E_{23210}]_{q^{(1-0)}} \text{ (since } E_{321}^2 = 0) \\ &\quad = (q+q^{-1}) \llbracket E_{321}, E_{23210} \rrbracket, \end{aligned}$$

$\llbracket \llbracket E_{321}, E_{23210} \rrbracket, F_2 K_2^{-1} \rrbracket = 0$ and $\llbracket E_{21}, F_2 K_2^{-1} \rrbracket = E_1$.

Using these, we have:

$$
\begin{aligned}
&[\![\mathcal{X}, F_2K_2^{-1}]\!] \\
&\quad = -q[(q+q^{-1})[\![E_{321}, E_{23210}]\!], E_1]_{q^{(-1-2)}} - (q+q^{-1})q^{-2}[E_1, [\![E_{321}, E_{23210}]\!]]_{q^{(2+1)}} \\
&\quad = 0
\end{aligned}
$$

Similarly we can show $[\![\mathcal{X}, F_1K_1^{-1}]\!] = [\![\mathcal{X}, F_0K_0^{-1}]\!] = 0$. By Proposition 6.2.1, it follows that $\mathcal{X} = 0 \in N^+$.

By similar calculation, we can get the relations (i)-(xix) of Proposition 6.3.1.

6.5. Let $(\mathcal{A}, \Delta)$ be a cocommutative Hopf C-superalgebra. Let

$$P(\mathcal{A}) = \{x \in \mathcal{A} | \Delta(X) = X \otimes 1 + 1 \otimes X\}.$$

Then $P(\mathcal{A})$ is a Lie C-superalgebra with a bracket $[\,,\,]$ given by $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$.

Let $\mathcal{G}$ be a Lie C-superalgebra and $U(\mathcal{G})$ its universal enveloping superalgebra. Then $U(\mathcal{G})$ is a cocommutative Hopf C-superalgebra with coproduct Δ such that $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in U(\mathcal{G})$. It is known that:

Theorem 6.5.1(Milnor-Moor [MM]) *Let $\mathcal{CSH}$ be the category of cocommutative Hopf C-superalgebras. and $\mathcal{SL}$ the category of Lie C-superalgebras. Define morphisms $\mathcal{P}$ and $\mathcal{U}$ by $\mathcal{P} : \mathcal{CSH} \to \mathcal{SL}$ ($\mathcal{A} \to P(\mathcal{A})$), $\mathcal{U} : \mathcal{SL} \to \mathcal{CSH}$ ($\mathcal{G} \to U(\mathcal{G})$). Then $\mathcal{PU} = id_{\mathcal{CSH}}$ and $\mathcal{UP} = id_{\mathcal{SL}}$.*

As an immediate consequence of Theorem 6.5.1, we have:

Lemma 6.5.1 *For a datum $(\mathcal{E}, \Pi, p)$, let $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$ and $U_0(\mathcal{G})$ a cocommutative Hopf C-superalgebra defined by $U_0(\mathcal{G}) = U_h(\mathcal{G})/hU_h(\mathcal{G})$. Then there is an epimorphism*

$$\phi : U_0(\mathcal{G}) \to U(\mathcal{G}) \quad (H,\, E_\alpha,\, F_\alpha \to H,\, E_\alpha,\, F_\alpha).$$

Proof. Let $\mathcal{G}_0 = P(U_0(\mathcal{G}))$. By theorem 6.5.1, we have $U_0(\mathcal{G}) = U(\mathcal{G}_0)$. Hence $\mathcal{G}_0$ should be a Lie C-superalgebra generated by H, E_α, F_α. By Theorem 6.1.1 (a), H, E_α, F_α satisfy (1.2.1-3). Since $U_h(\mathcal{G})$ has the triangular decomposition by Theorem 6.1.1 (b), $U_0(\mathcal{G})$ also has a triangular decomposition. In particular, $\mathcal{H}$ can be embedded into $\mathcal{G}_0(\subset U_0(\mathcal{G}))$. By definition of the Kac-Moody Lie superalgebra $\mathcal{G}$ (see [K1]), we have epimorphism $\phi_{|\mathcal{G}_0} : \mathcal{G}_0 \to \mathcal{G}$ (H, E_α, $F_\alpha \to H$, E_α, F_α). Hence we have ϕ.

Q.E.D.

6.6. Theorem 6.6.1 *Let $(\mathcal{E}, \Pi, p)$ be a datum of affine type. Assume $\bar{\mathcal{G}}(\mathcal{E}, \Pi, p) = \mathcal{G}(\mathcal{E}, \Pi, p)$,i.e., $\mathcal{G}(\mathcal{E}, \Pi, p)$ is not of $\hat{s}l(m|m)^{(i)}$ $(i = 1, 2, 4)$. Put $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$. Then the defining relations of $U_h(\mathcal{G})$ are given by*
(i) The relations of Theorem 6.1.1 (a),
(ii) The relations of Proposition 6.3.1 (i)-(xix),
(iii) The same relations of F_α's as (ii).

Proof. By Theorem 6.1.1, Proposition 6.3.1, $U_h(\mathcal{G})$ satisfies the relations (i)-(iii). Therefore, by Serre type theorems of Chapters 4 and 5, we have an epimorphism $\psi : U(\mathcal{G}) \to U_0(\mathcal{G})$ $(H,\ E_\alpha,\ F_\alpha \to H,\ E_\alpha,\ F_\alpha)$. By Lemma 6.5.1, ψ should be the inverse map of ϕ. Then $U_0(\mathcal{G}) = U(\mathcal{G})$. The relations given by putting $h = 0$ on (i)-(iii) are the defining relations of $U(\mathcal{G})$. Hence, by the topologically freedom of $U_h(\mathcal{G})$, the relations (i)-(iii) should be the defining relations of $U_h(\mathcal{G})$.

Q.E.D.

6.7 Lemma 6.7.1. *Let $U_h(\dot{\mathcal{G}})$ be the topologically free Hopf C[[h]]-superalgebra with generators $\{H, E_\alpha, F_\alpha\}$ $(\alpha \in \Pi)$ such that*

(1) $\{H, E_\alpha, F_\alpha\}$ satisfy (6.1.3-4).
(2) The map $\mathcal{H} \to U_h(\dot{\mathcal{G}})$ $(H \to H)$ is injective.

Then there is a Hopf superalgebra epimorphism

$$\dot{J} : U_h(\dot{\mathcal{G}}) \to U_h(\mathcal{G}) \quad (H, E_\alpha, F_\alpha \to H, E_\alpha, F_\alpha).$$

Proof. By (2), the topological freedom of $U_h(\dot{\mathcal{G}})$ and Theorem 6.5.1, $C[[h]]\,[\mathcal{H}]$ is embedded into $U_h(\dot{\mathcal{G}})$. Let $U_h(\dot{\mathcal{N}}^+)$, $U_h(\dot{\mathcal{N}}^-)$ be non-topological subalgebras generated by E_α, F_α respectively. Put $U_0(\dot{\mathcal{G}}) = U_h(\dot{\mathcal{G}})/hU_h(\dot{\mathcal{G}})$ and $U_0(\dot{\mathcal{N}}^\pm) = U_h(\dot{\mathcal{N}}^\pm)/hU_h(\dot{\mathcal{N}}^\pm)$. By Minor-Moor's Theorem 6.5.1, $U_0(\dot{\mathcal{G}})$ (resp. $U_0(\dot{\mathcal{N}}^\pm)$) is the universal enveloping algebra $U(\dot{\mathcal{G}})$ (resp. $U(\dot{\mathcal{N}}^\pm)$) of a Lie C-superalgebra $\dot{\mathcal{G}} = \mathcal{P}(U_0(\dot{\mathcal{G}}))$. (resp. $\dot{\mathcal{N}}^\pm = \mathcal{P}(U_0(\dot{\mathcal{N}}^\pm))$ By (2), $\mathcal{H}$ is embedded into $\dot{\mathcal{G}}$. Hence we have the triangular decompositions $\dot{\mathcal{G}} = \dot{\mathcal{N}}^- \oplus \mathcal{H} \oplus \dot{\mathcal{N}}^+$ and $U(\dot{\mathcal{G}}) = U(\dot{\mathcal{N}}^-) \otimes U(\mathcal{H}) \oplus U(\dot{\mathcal{N}}^+)$. By the topological freedom of $U_h(\dot{\mathcal{G}})$, we have the triangular decomposition $U_h(\dot{\mathcal{G}}) = U_h(\dot{\mathcal{N}}^-)\hat{\otimes}C[[h]]\,[\mathcal{H}]\,\hat{\otimes}U_h(\dot{\mathcal{N}}^+)$.

Let $\dot{I}^+ = \ker(\tilde{N}^+ \to \dot{\mathcal{N}}^+)$. For $\gamma \in P_+$, put $\tilde{N}^+_\gamma$ (resp. $\dot{I}^+_\gamma) = \{X \in \tilde{N}^+ \,(\text{resp } \dot{I})\,|[H_\lambda, X] = (\lambda, \gamma)X\}$. Then we have $\tilde{N}^+ = \oplus_{\gamma \in P_+}\tilde{N}^+_\gamma$ and $\dot{I}^+ = \oplus_{\gamma \in P_+}\dot{I}^+_\gamma$. *Keep notations in 6.2.* Since $\dot{I}^+$ is an ideal of $\tilde{N}^+$, by the triangular decomposition of $U_h(\dot{\mathcal{G}})$,

$$e([\dot{I}^+_\gamma, F_{\alpha(1)} \cdots F_{\alpha(r)}]) = 0 \quad \text{for} \sum \alpha(i) = \gamma \ (\alpha(i) \in \Pi).$$

Hence, by a $U_h(\dot{\mathcal{G}})$-version of Lemma 6.2.2, since $\tilde{N}^+ = \oplus_{\gamma\in P_+}\tilde{N}^+_\gamma$ is orthogonal with respect to $\langle\, ,\, \rangle$, we have $\dot{I}^+_\gamma \subset I^+ = \ker\langle\, ,\, \rangle$. Then we have the algebra epimorphism $U_h(\dot{\mathcal{N}}^+) \to N^+$. $(E_\alpha \to E_\alpha)$.

Next we should show the existence of the epimorphism $\dot{N}^- \to N^-$. However a Hopf superalgebra $(U_h(\dot{\mathcal{G}}), s\tau\circ\Delta, S^{-1}, \varepsilon)$ with generators $\{-H_\lambda, F_\alpha, (-1)^{p(\alpha)}E_\alpha\}$ also satisfies (1) and (2). (Here $s\tau(X\otimes Y) = (-1)^{p(X)p(Y)}Y\otimes X$). Then the same argument can be applied for the subalgebra generated by F_α. Hence we can show the existence.

Eventually we get a $C[[h]]$-module surjective map:

$$\dot{J} : U_h(\dot{\mathcal{G}}) = (\dot{N}^- \otimes C[[h]]\,[\mathcal{H}] \otimes \dot{N}^+)^\wedge U_h(\mathcal{G}) = (N^- \otimes C[[h]]\,[\mathcal{H}] \otimes N^+)^\wedge.$$

Considering under the two triangular decompositions, it is clear that $\dot{J}$ preserve product. Hence $\dot{J}$ is the algebra epimorphism. Clearly $\dot{J}$ is the Hopf superalgebra epimorphism.

Q.E.D.

7. Quantization of Weyl-group-type isomorphisms

7.1 Let $(\mathcal{C}, \Delta, S, \varepsilon)$ be a topological Hopf $C[[h]]$-algebra. Define $\tau : \mathcal{C}\hat{\otimes}\mathcal{C} \to \mathcal{C}\hat{\otimes}\mathcal{C}$ by $\tau(x\otimes y) = y\otimes x$. Let $\Delta' = \tau\circ\Delta$. Let $\mathcal{C}_0$ be a Hopf subalgebra of $\mathcal{C}$. Let $R = \sum a_i \otimes b_i$ be an invertible element of $\mathcal{C}_0\otimes\mathcal{C}_0$ satisfying:

$$R\Delta(x)R^{-1} = \Delta'(x), \tag{7.1.1}$$
$$(\Delta\otimes I)(R) = R_{13}R_{23},\ \ (I\otimes\Delta)(R) = R_{13}R_{12}. \tag{7.1.2}$$

where $R_{12} = R\otimes I$, $R_{23} = I\otimes R$ and $R_{13} = \sum a_i\otimes I\otimes b_i$.

By Drinfeld[D2], we have known:

Proposition 7.1.1.(Drinfeld[D2]) *(i) R satisfies:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{13}, \tag{7.1.3}$$
$$(S\otimes I)(R) = R^{-1} = (I\otimes S^{-1})(R), \tag{7.1.4}$$
$$(\varepsilon\otimes I)(R) = 1 = (I\otimes\varepsilon)(R). \tag{7.1.5}$$

(ii) For $R = \sum a_i\otimes b_i$, following equations hold in $\mathcal{C}$:

$$\sum a_i S^{-2}(b_i) = \sum S(a_i)S^{-1}(b_i) = \sum S^2(a_i)b_i, \tag{7.1.6}$$
$$\sum a_i S(b_i) = \sum S^{-1}(a_i)b_i. \tag{7.1.7}$$

Let u_4, $v_4\in\mathcal{C}$ be the elements of (7.1.6), (7.1.7) respectively. Then $u_4v_4 = 1 = v_4u_4$.

7.2. Proposition 7.2.1. *Keep notations in 7.1. For the Hopf algebra $\mathcal{C}$*

and the element $R = \sum a_i \otimes b_i$ satisfying (7.1.1-2), there is an another Hopf algebra structure $(\mathcal{C}^{(R)}) = (\mathcal{C}, \Delta^{(R)}, S^{(R)}, \varepsilon)$ given by:

$$\Delta^{(R)}(x) = R\Delta(x)R^{-1},\ S^{(R)}(x) = u_4^{-1}S(x)u_4.$$

Proof. First we show $(I \otimes \Delta^{(R)}) \circ \Delta^{(R)} = (\Delta^{(R)} \otimes I) \circ \Delta^{(R)}$. By (7.1.1-2), We have:

$$\begin{aligned}
&(I \otimes \Delta^{(R)}) \circ \Delta^{(R)}(x) \\
&\quad = R_{23}(I \otimes \Delta)(R\Delta(x)R^{-1})R_{23}^{-1} \\
&\quad = R_{23}R_{13}R_{12}(I \otimes \Delta)(\Delta(x))R_{12}^{-1}R_{13}^{-1}R_{23}^{-1} \\
&\quad = R_{12}R_{13}R_{23}(\Delta \otimes I)(\Delta(x))R_{23}^{-1}R_{13}^{-1}R_{12}^{-1} \\
&\quad = (\Delta^{(R)} \otimes I) \circ \Delta^{(R)}(x).
\end{aligned}$$

Let $m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ be the multiplication, which is defined by $m(x \otimes y) = xy$. Next we show $m \circ (I \otimes S^{(R)}) \circ \Delta^{(R)} = \varepsilon = m \circ (S^{(R)} \otimes I) \circ \Delta^{(R)}$. By (7.1.4) and (7.1.6-7), for $x \in \mathcal{C}$ with $\Delta(x) = \sum x_i^{(1)} \otimes x_i^{(2)}$, we have:

$$\begin{aligned}
&(I \otimes S^{(R)}) \circ \Delta^{(R)}(x) \\
&\quad = \sum m((1 \otimes u_4^{-1})(I \otimes S)(a_i x_j^{(1)} S(a_l) \otimes b_i x_j^{(2)} b_l)(1 \otimes u_4)) \\
&\quad = \sum a_i x_j^{(1)} S(a_l) \cdot a_y S(b_l) S(b_i) S(x_j^{(2)}) S(b_i) u_4 \\
&\quad = \sum a_i x_j^{(1)} S(x_j^{(2)}) S(b_i) u_4 \quad \text{since } (S \otimes I)(R)R = 1 \\
&\quad = \varepsilon(x) \sum a_i S(b_i) u_4 = \varepsilon(x) v_4 u_4 = \varepsilon(x)
\end{aligned}$$

and

$$\begin{aligned}
&(S^{(R)} \otimes I) \circ \Delta^{(R)}(x) \\
&\quad = \sum m((u_4^{-1} \otimes 1)(S \otimes I)(a_i x_j^{(1)} S(a_l) \otimes b_i x_j^{(2)} b_l)(u_4 \otimes 1)) \\
&\quad = \sum u_4^{-1} S^2(a_l) S(x_j^{(1)}) S(a_i) S(a_y) S^{-1}(b_y) \cdot b_i x_j^{(2)} b_l \\
&\quad = \sum u_4^{-1} S^2(a_l) S(x_j^{(1)}) x_j^{(2)} b_l \quad \text{since } (I \otimes S^{-1})(R)R = 1 \\
&\quad = \varepsilon(x) u_4^{-1} \sum S^2(a_l) b_l = \varepsilon(x) u_4^{-1} u_4 = \varepsilon(x).
\end{aligned}$$

To show other formulae of the axiom of the Hopf algebra are easy.

Q.E.D.

7.3. Let $(\mathcal{E}, \Pi, p)$ be a datum and $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$. Let $U_h(\mathcal{G})$ be a topological Hopf $C[[h]]$-superalgebra introduced in 6.1. Put $U_h(\mathcal{G})^\sigma = U_h(\mathcal{G}) \otimes_{C[[h]]}$

$C[[h]]\langle\sigma\rangle$. Then $U_h(\mathcal{G})^\sigma$ is an algebra with a formula $\sigma X\sigma = (-1)^{p(X)}X$ $(X \in U_h(\mathcal{G}))$. By [Y1], $(U_h(\mathcal{G})^\sigma, \Delta, S, \varepsilon)$ is a Hopf algebra such that

$$\begin{aligned}
&\Delta(H) = H\otimes 1 + 1\otimes H,\ \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha\sigma^{p(\alpha)}\otimes E_\alpha,\\
&\Delta(F_\alpha) = F_\alpha\otimes K_\alpha^{-1} + \sigma^{p(\alpha)}\otimes F_\alpha,\\
&S(H) = -H,\ S(E_\alpha) = -K_\alpha^{-1}\sigma^{p(\alpha)}E_\alpha,\ S(F_\alpha) = -\sigma^{p(\alpha)}F_\alpha K_\alpha\\
&\varepsilon(H) = \varepsilon(E_\alpha) = \varepsilon(F_\alpha) = 0.
\end{aligned}$$

Put $U_h(\mathcal{H})^\sigma = C[[h]][\mathcal{H}]\otimes C[[h]]\langle\sigma\rangle$. Then $U_h(\mathcal{H})^\sigma$ is a Hopf subalgebra of $U_h(\mathcal{G})^\sigma$. Put $t_0 = \sum H_{\delta_i}\otimes H_{\delta_i} \in \mathcal{H}\otimes\mathcal{H}$ where $\{\delta_i\}$ is a C-basis of $\mathcal{H}$ such that $(\delta_i,\delta_j) = \delta_{ij}$. Then, by the quantum double construction (see [D] (also [Y1])),

$$R_T = \frac{1}{2}\Big(\sum_{c,d=0,1}(-1)^{cd}\sigma^c\otimes\sigma^d\Big)\cdot\exp(-ht_0) \in U_h(\mathcal{H})^\sigma\otimes U_h(\mathcal{H})^\sigma$$

satisfies (7.1.1-2). Clearly R_T^{-1} also satisfies (7.1.1-2).

For $t \in C[[h]]$ and $n > 0$, we put $\{n\}_t = \frac{t^n-1}{t-1}$, $\{n\}_t! = \{n\}_t\{n-1\}_t\cdots\{1\}_t$ and

$$\begin{Bmatrix} n \\ m\end{Bmatrix}_t = \begin{cases} \frac{\{n\}_t!}{\{m\}_t!\{n-m\}_t!} & \text{if } n\le m\le 0,\\ 0 & \text{otherwise.}\end{cases}$$

For $u \in hU_h(\mathcal{G})^\sigma$, put $e(u,t) = \displaystyle\sum_{n=0}^{\infty}\frac{u^n}{\{n\}_t!}$. It is easy to show that

$$e(-u,t^{-1}) = e(u,t)^{-1}, \tag{7.3.1}$$
$$e(u,t)Xe(u,t)^{-1} = \sum_{n=0}^{\infty}\frac{1}{\{n\}_t!}ad_{t^{n-1}}(u)ad_{t^{n-2}}(u)\cdots ad_1(u)(X) \tag{7.3.2}$$

where $ad_x(u)(X) = [u,X]_{-,x} = uX - xXu$.

For $\alpha\in\Pi$, let $U_h(\mathcal{G}^{(\alpha)})^\sigma$ be a topological subalgebra of $U_h(\mathcal{G})^\sigma$ generated by $U_h(\mathcal{H})^\sigma$ and E_α, F_α. By the quantum double construction, we see that

$$R_\alpha = e(-(q-q^{-1})E_\alpha\otimes F_\alpha\sigma^{p(\alpha)}, (-1)^{p(\alpha)}q^{(\alpha,\alpha)})\cdot R_T \in U_h(\mathcal{G}^{(\alpha)})^\sigma\otimes U_h(\mathcal{G}^{(\alpha)})^\sigma$$

satisfies (7.1.1-2). Let $(U_h(\mathcal{G})^\sigma)^{(\alpha)} = (U_h(\mathcal{G})^\sigma, \Delta^{(\alpha)}, S^{(\alpha)}, \varepsilon)$ be an another Hopf algebra defined as $((U_h(\mathcal{G})^\sigma)^{(R_\alpha)})^{(R_T^{-1})}$. Put

$$\hat{R}_\alpha = e(-(q-q^{-1})\sigma^{p(\alpha)}K_\alpha^{-1}E_\alpha\otimes F_\alpha K_\alpha, (-1)^{p(\alpha)}q^{(\alpha,\alpha)}).$$

Then we get $\hat{R}_\alpha = R_T^{-1}R_\alpha$. Hence

$$\Delta^{(\alpha)}(X) = \hat{R}_\alpha\Delta(X)\hat{R}_\alpha^{-1}\quad (X\in U_h(\mathcal{G})^\sigma).$$

Proposition 7.3.1. *For $\alpha,\beta\in\Pi$. Put $E^\vee_{\beta+s\alpha} = [\![...[\![[\![E_\beta, E_\alpha]\!], E_\alpha]\!]\ldots E_\alpha]\!]$, $F^\vee_{\beta+s\alpha} = [\![...[\![[\![F_\beta, F_\alpha]\!], F_\alpha]\!]\ldots F_\alpha]\!]$ (E_α, F_α appears s-times).*

(i) $\quad\Delta^{(\alpha)}(E_\alpha K_\alpha^{-1}) = E_\alpha K_\alpha^{-1}\otimes K_\alpha + \sigma^{p(\alpha)}\otimes E_\alpha K_\alpha^{-1},$

$$\Delta^{(\alpha)}(K_\alpha F_\alpha) = K_\alpha F_\alpha \otimes 1 + \sigma^{p(\alpha)} K_\alpha^{-1} \otimes K_\alpha F_\alpha.$$

(ii) Assume $(\alpha,\alpha) \neq 0$. *For* $\beta \in \Pi$, *assume* $r = r_{(\alpha,\beta)} = -\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in Z_+$ *and* $p(\alpha)\cdot r$ *is even. Then*

$$\Delta^{(\alpha)}(E^\vee_{\beta+r\alpha}) = E^\vee_{\beta+r\alpha} \otimes 1 + K_{\beta+r\alpha}\sigma^{p(\beta+r\alpha)} \otimes E^\vee_{\beta+r\alpha},$$
$$\Delta^{(\alpha)}(F^\vee_{\beta+r\alpha}) = F^\vee_{\beta+r\alpha} \otimes K^{-1}_{\beta+r\alpha} + \sigma^{p(\beta+r\alpha)} \otimes F^\vee_{\beta+r\alpha}.$$

(iii) Assume $(\alpha,\alpha) = 0$ *and* $(\alpha,\beta) \neq 0$. *Then*

$$\Delta^{(\alpha)}(E^\vee_{\beta+\alpha}) = E^\vee_{\beta+\alpha} \otimes 1 + K_{\beta+\alpha}\sigma^{p(\beta+\alpha)} \otimes E^\vee_{\beta+\alpha},$$
$$\Delta^{(\alpha)}(F^\vee_{\beta+\alpha}) = F^\vee_{\beta+\alpha} \otimes K^{-1}_{\beta+\alpha} + \sigma^{p(\beta+\alpha)} \otimes F^\vee_{\beta+\alpha}.$$

(iv) $\Delta^{(\alpha)}(H) = H \otimes 1 + 1 \otimes H$, $\Delta^{(\alpha)}(\sigma) = \sigma \otimes \sigma$.

Proof. Here we calculate $\Delta^{(\alpha)}(E^\vee_{\beta+r\alpha})$ of (ii). Put $t_\alpha = (-1)^{p(\alpha)} q^{(\alpha,\alpha)}$ and $t_{\alpha,\beta} = (-1)^{p(\alpha)p(\beta)} q^{(\alpha,\beta)}$. By direct calculation, we have:

$$\Delta(E^\vee_{\beta+u\alpha}) = E^\vee_{\beta+u\alpha} \otimes 1 + \sum_{s=0}^{u} \begin{Bmatrix} u \\ s \end{Bmatrix}_{t_\alpha} \prod_{k=1}^{u-s} (t^{-1}_{\alpha,\beta} - t^{-1}_{\alpha,\beta} t_\alpha^{k-u}) E_\alpha^{u-s} K_{\beta+s\alpha}\sigma^{p(\beta+s\alpha)} \otimes E^\vee_{\beta+s\alpha}.$$

Hence, for r of (ii),

$$\Delta(E^\vee_{\beta+r\alpha}) = E^\vee_{\beta+r\alpha} \otimes 1 + \sum_{s=0}^{r} \begin{Bmatrix} r \\ s \end{Bmatrix}_{t_\alpha} t^{r-s}_{\alpha,\beta} \prod_{k=1}^{r-s} (1 - t_\alpha^{k}) E_\alpha^{r-s} K_{\beta+s\alpha}\sigma^{p(\beta+s\alpha)} \otimes E^\vee_{\beta+s\alpha}.$$

Put $X = -\sigma^{p(\alpha)} K_\alpha^{-1} \otimes F_\alpha K_\alpha$. Then

$$[-X, E_\alpha^s K_{\beta+(r-s)\alpha}\sigma^{p(\beta+(r-s)\alpha)} \otimes E^\vee_{\beta+(r-s)\alpha}]_{-,t^{-s}}$$
$$= \begin{cases} -\frac{t_{\alpha,\beta}}{q-q^{-1}} t_\alpha^{-s} \frac{(t_\alpha^{r-s}-1)(t_\alpha^{s+1}-1)}{t_\alpha - 1} E_\alpha^{s+1} K_{\beta+(r-s-1)\alpha}\sigma^{p(\beta+(r-s-1)\alpha)} \otimes E^\vee_{\beta+(r-s-1)\alpha} \\ \quad \text{if } r > s, \\ 0 \quad \text{if } r = s. \end{cases}$$

Hence

$$ad_{t_\alpha^{-(s-1)}}(X) ad_{t_\alpha^{-(s-2)}}(X) \cdots ad_1(X)(K_{\beta+r\alpha}\sigma^{p(\beta+r\alpha)} \otimes E^\vee_{\beta+r\alpha})$$
$$= \begin{cases} = \left(\frac{t_{\alpha,\beta}}{q-q^{-1}}\right)^s t_\alpha^{-\frac{s(s-1)}{2}} (t_\alpha - 1)^s \begin{Bmatrix} r \\ s \end{Bmatrix}_{t_\alpha} E_\alpha^s K_{\beta+(r-s)\alpha}\sigma^{p(\beta+(r-s)\alpha)} \otimes E^\vee_{\beta+(r-s)\alpha} \\ \quad \text{if } r \geq s, \\ 0 \quad \text{if } r < s. \end{cases}$$

Hence, by (7.3.1-2),

$$\hat{R}_\alpha^{-1}(K_{\beta+r\alpha}\sigma^{p(\beta+r\alpha)} \otimes E^\vee_{\beta+r\alpha})\hat{R}_\alpha = \sum_{s=0}^{r} (-t_\alpha)^s (t-1)^s \frac{\{r\}_{t_\alpha}!}{\{r-s\}_{t_\alpha}!} E_\alpha^s K_{\beta+(r-s)\alpha}\sigma^{p(\beta+(r-s)\alpha)} \otimes E^\vee_{\beta+(r-s)\alpha}.$$

On the other hand, we can easily show $\hat{R}_\alpha^{-1}(E^\vee_{\beta+r\alpha}\otimes 1)\hat{R}_\alpha = E^\vee_{\beta+r\alpha}\otimes 1$. Then we get:

$$\hat{R}_\alpha^{-1}(E^\vee_{\beta+r\alpha}\otimes 1 + K_{\beta+r\alpha}\sigma^{p(\beta+r\alpha)}\otimes E^\vee_{\beta+r\alpha})\hat{R}_\alpha = \Delta(E^\vee_{\beta+r\alpha})\,.$$

We can show other formulae similarly or easily.

Q.E.D.

By [Y1], we see:

Lemma 7.3.1. *For $\alpha\in\Pi$, $U_h(\mathcal{G})$ has an another Hopf superalgebra structure $U_h(\mathcal{G})^{(\alpha)} = (U_h(\mathcal{G}),\Delta_s^{(\alpha)})$ with coproduct $\Delta_s^{(\alpha)}$ satisfies formulae given by eliminating $\sigma^{p(\cdot)}$ in the formulae of $\Delta^{(\alpha)}$ of Proposition 7.3.1 (i)-(iv).*

7.4. Lemma 7.4.1. *Keep notation in 7.3. Let $\alpha\in\Pi$.*
(i) Assume $(\alpha,\alpha)\neq 0$. Assume $r = r_{\alpha,\beta}\in Z_+$ and $p(\alpha)r\in 2Z$ for any $\beta\in\Pi\setminus\{\alpha\}$. Define $\sigma_\alpha:\mathcal{H}\to\mathcal{H}$ by $\sigma_\alpha(H_\lambda) = H_{\lambda-\frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha}$. Let $x_\beta, y_\beta\in C[[h]]^\times$ $(\beta\in\Pi)$ be such that

$$x_\beta y_\beta = \begin{cases} (-1)^{p(\beta)} & (\beta=\alpha),\\ (-1)^{r_{\alpha,\beta}}q^{-(r_{\alpha,\beta}+1)(\alpha,\beta)}\left(\frac{1-t_\alpha^{-1}}{q-q^{-1}}\right)^{r_{\alpha,\beta}}(\{r_{\alpha,\beta}\}_{t_\alpha^{-1}}!)^2 & (\beta\neq\alpha,\,(\alpha,\beta)\neq 0),\\ 1 & (\beta\neq\alpha,\,(\alpha,\beta)=0).\end{cases}$$

Put $H'_\lambda = H_{\sigma_\alpha(\lambda)}$, $E'_\alpha = x_\alpha^{-1}F_\alpha K_\alpha$, $F'_\alpha = y_\alpha^{-1}K_\alpha^{-1}E_\alpha$, $E'_\beta = x_\beta^{-1}E^\vee_{\beta+r_{\alpha,\beta}\alpha}$, $F'_\beta = y_\beta^{-1}F^\vee_{\beta+r_{\alpha,\beta}\alpha}$ $(\beta\in\Pi\setminus\{\alpha\})$.
Then H'_λ, E'_β, F'_β satisfy (6.1.3).

(ii) Assume $(\alpha,\alpha)=0$. For $\beta\in\Pi\setminus\{\alpha\}$, put

$$r_{\alpha,\beta} = \begin{cases} 1 & (\alpha,\beta)\neq 0\\ 0 & (\alpha,\beta)=0\end{cases}$$

Let $\Pi' = \{\alpha' = -\alpha,\ \beta' = \alpha+\beta\,(\beta\in\Pi,\,(\alpha,\beta)\neq 0),\ \gamma'=\gamma\,(\gamma\in\Pi\setminus\{\alpha\},\,(\gamma,\alpha)=0)\}$. Let $\sigma_\alpha:(\mathcal{E},\Pi,p)\to(\mathcal{E},\Pi',p)$ by $\sigma_\alpha(H)=H$. (In particular,

$$\sigma_\alpha(H_\beta) = \begin{cases} H_{-\alpha'} & (\beta=\alpha),\\ H_{\beta'+r_{\alpha,\beta}\alpha'} & (\beta\neq\alpha,\,(\alpha,\beta)\neq 0).)\end{cases}$$

Let $x_\beta, y_\beta\in C[[h]]^\times$ $(\beta\in\Pi)$ be such that

$$x_\beta y_\beta = \begin{cases} -1 & (\beta=\alpha),\\ t_{\alpha,\beta}^{-1}\frac{q^{(\alpha,\beta)}-q^{-(\alpha,\beta)}}{q-q^{-1}} & (\beta\neq\alpha,\,(\alpha,\beta)\neq 0),\\ 1 & (\beta\neq\alpha,\,(\alpha,\beta)=0).\end{cases}$$

Put $E'_\alpha = x_\alpha^{-1}F_\alpha K_\alpha$, $F'_\alpha = y_\alpha^{-1}K_\alpha^{-1}E_\alpha$, $E'_\beta = x_\beta^{-1}E^\vee_{\beta+r_{\alpha,\beta}\alpha}$, $F'_\beta = y_\beta^{-1}F^\vee_{\beta+r_{\alpha,\beta}\alpha}$ $(\beta \in \Pi\setminus\{\alpha\})$.

Then H, E'_β, F'_β *satisfy (6.1.3) for* $(\mathcal{E},\Pi',p)$.

Proof. Here we show how to calculate

$$[E^\vee_{\beta+r_{\alpha,\beta}\alpha}, F^\vee_{\beta+r_{\alpha,\beta}\alpha}] = x_\beta y_\beta \frac{\sinh(hH_{\beta+r_{\alpha,\beta}\alpha})}{\sinh(h)}. \tag{7.4.1}$$

Put $\{k;\beta\}_\alpha = \frac{q^{-(\alpha,\beta)}(1-t_\alpha^{-k})(1-t^2_{\alpha,\beta}t_\alpha^{k-1})}{(q-q^{-1})(1-t_\alpha^{-1})}$ and $\{k;\beta\}_\alpha! = \prod_{v=1}^{k}\{v;\beta\}_\alpha$. First we show:

$$[E_\alpha, F^\vee_{\beta+k\alpha}] = -t^{-1}_{\alpha,\beta}q^{-(k-1)(\alpha,\alpha)}\{k;\beta\}_\alpha F^\vee_{\beta+(k-1)\alpha}K_\alpha\,,$$
$$[E^\vee_{\beta+k\alpha}, F_\alpha] = (-1)^{(k-1)p(\alpha)}\{k;\beta\}_\alpha K_\alpha^{-1}E^\vee_{\beta+(k-1)\alpha}\,.$$

Then, by induction on k, we can show:

$$\begin{aligned}&[E^\vee_{\beta+k\alpha}, F^\vee_{\beta+(k-1)\alpha}]\\ &\quad = (-1)^{k(1+p(\alpha)p(\beta))}q^{-\frac{(k-1)(k-2)}{2}(\alpha,\alpha)}q^{(-k+1)(\alpha,\beta)}\{k;\beta\}_\alpha! E_\alpha K_{\beta+(k-1)\alpha},\end{aligned}$$

$$\begin{aligned}&[E^\vee_{\beta+(k-1)\alpha}, F^\vee_{\beta+k\alpha}]\\ &\quad = (-1)^{(k-1)(1+p(\alpha)+p(\alpha)p(\beta))}q^{-\frac{k(k-1)}{2}(\alpha,\alpha)}q^{-k(\alpha,\beta)}\{k;\beta\}_\alpha! K^{-1}_{\beta+(k-1)\alpha}F_\alpha\end{aligned}$$

and

$$\begin{aligned}&[E^\vee_{\beta+k\alpha}, F^\vee_{\beta+k\alpha}]\\ &\quad = (-1)^k t^{-k}_{\alpha,\beta}q^{-\frac{k(k-1)}{2}(\alpha,\alpha)}\{k;\beta\}_\alpha! \frac{\sinh(hH_{\beta+k\alpha})}{\sinh(h)}.\end{aligned}$$

Substituting $r_{\alpha,\beta}$ for k, we get (7.3.1).

We can show other formulae similarly or easily.

Q.E.D.

7.5. Proposition 7.5.1. *Keep notations in 7.4.*

(i) Let $\Pi^{\sigma_\alpha} = \Pi$ *if* $(\alpha,\alpha)\neq 0$ *and let* $\Pi^{\sigma_\alpha} = \Pi'$ *if* $(\alpha,\alpha) = 0$. *Put* $U_h(\mathcal{G}^{\sigma_\alpha}) = U_h(\mathcal{G}(\mathcal{E},\Pi^{\sigma_\alpha},p))$. *Then there is an isomorphism* $L_\alpha : U_h(\mathcal{G}) \to U_h(\mathcal{G}^{\sigma_\alpha})$ *such that*

$$L_\alpha(H) = \sigma_\alpha(H),\ L_\alpha(E_\beta) = E'_\beta,\ L_\alpha(F_\beta) = F'_\beta. \tag{7.5.1}$$

(ii)

$$\Delta(L_\alpha(X)) = \hat{R}_\alpha^{-1}(L_\alpha\otimes L_\alpha\Delta(X))\hat{R}_\alpha \quad (X\in U_h(\mathcal{G})^\sigma). \tag{7.5.2}$$

Proof. (i) By Lemma 6.7.1, Lemma 7.3.1 and Lemma 7.4.1, there is an epimorphism $L'_\alpha : U_h(\mathcal{G}) \to U_h(\mathcal{G}^{\sigma_\alpha})$ satisfying (7.5.1). Let $L_\alpha : U_h(\mathcal{G}) \to U_h(\mathcal{G}^{\sigma_\alpha})$ denote L'_α defined by changing $U_h(\mathcal{G})$ and $U_h(\mathcal{G}^{\sigma_\alpha})$. (*Keep notations*

in the proof of Lemma 6.5.1.) Since $L'_\alpha L_{\alpha|\mathcal{H}} = id_{\mathcal{H}}$ and (resp. $L_\alpha L'_{\alpha|\mathcal{H}} = id_{\mathcal{H}}$), $L'_\alpha L_\alpha$ (resp. $L_\alpha L'_\alpha$) induce an automorphism of $\mathcal{G}_0$ (resp. $\mathcal{G}_0^{\sigma_\alpha}$) as well as an automorphism of $U_0(\mathcal{G}_0)$ (resp. $U_0(\mathcal{G}_0^{\sigma_\alpha})$). Hence L_α induce an isomorphism $U_0(\mathcal{G}_0) \to U_0(\mathcal{G}_0^{\sigma_\alpha})$. Hence by topological freedom of $U_h(\mathcal{G})$ and $U_h(\mathcal{G}^{\sigma_\alpha})$, L_α is an isomorphism.
(ii) (7.5.2) is clear from the formulae in Proposition 7.3.1 (i)-(iii).

Q.E.D.

By direct calculation, we have:

Lemma 7.5.1. *$L_\alpha^{-1} : U_h(\mathcal{G}) \to U_h(\mathcal{G}^{\sigma_\alpha})$ satisfies (Here we let the region of definition (resp. values) of L_α^{-1} be $(\mathcal{E}, \Pi, p)$ (resp. $(\mathcal{E}, \Pi^{\sigma_\alpha}, p)$)). For $\alpha, \beta \in \Pi$. Put $E_{\beta+s\alpha} = [\![E_\alpha \dots [\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!]...]\!]$, $F_{\beta+s\alpha} = [\![F_\alpha \dots [\![F_\alpha, [\![F_\alpha, F_\beta]\!]]\!]...]\!]$, ($E_\alpha$, F_α appears s-times).*

Put $H''_\lambda = H_{\sigma_\alpha(\lambda)}$, $E''_\alpha = \dot{x}_\alpha^{-1} K_\alpha^{-1} F_\alpha$, $F''_\alpha = \dot{y}_\alpha^{-1} E_\alpha K_\alpha$, $E''_\beta = \dot{x}_\beta^{-1} E_{\beta + r_{\alpha,\beta}\alpha}$, $F''_\beta = \dot{y}_\beta^{-1} F_{\beta + r_{\alpha,\beta}\alpha}$ ($\beta \in \Pi \setminus \{\alpha\}$). Here we define $\dot{x}_\beta, \dot{y}_\beta \in C[[h]]^\times$ by:

$$
\begin{aligned}
&x_\alpha \dot{y}_\alpha = y_\alpha \dot{x}_\alpha = 1,\\
&y_\alpha^{r_{\alpha,\beta}} y_\beta \dot{y}_\beta = (-1)^{r_{\alpha,\beta}} (-1)^{(p(\alpha)+p(\alpha)p(\beta))r_{\alpha,\beta}} \{r_{\alpha,\beta}; \beta\}_\alpha !,\\
&x_\alpha^{r_{\alpha,\beta}} x_\beta \dot{x}_\beta = (-1)^{r_{\alpha,\beta}} (-1)^{p(\alpha)p(\beta)r_{\alpha,\beta}} q^{r_{\alpha,\beta}(\alpha,\alpha)} \{r_{\alpha,\beta}; \beta\}_\alpha ! \quad (\beta \neq \alpha,\, (\alpha, \beta) \neq 0),\\
&x_\beta \dot{x}_\beta = y_\beta \dot{x}_\beta = 1 \quad (\beta \neq \alpha,\, (\alpha, \beta) = 0).
\end{aligned}
$$

(Here $x_\beta, y_\beta \in C[[h]]^\times$ have been defined in Lemma 7.4.1 for $L_\alpha : U_h(\mathcal{G}^{\sigma_\alpha}) \to U_h(\mathcal{G})$.)

7.6. As an immediate consequence of Proposition 7.5.1, we have:

Proposition 7.6.1. *(See also [KT].) Let $(\mathcal{E}, \Pi, p)$'s be the data of affine type. For the isomorphisms L_i defined for $\mathcal{G}(\mathcal{E}, \Pi, p)$'s in §2, there are isomorphisms T_i's of $U_h(\mathcal{G}(\mathcal{E}, \Pi, p))$'s such that $T_i \to L_i : U_0(\mathcal{G}(\mathcal{E}, \Pi, p)) \to U_0(\mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)}))$ ($h \to 0$).*

8. On $U_h(\hat{sl}(m|m))^{(i)}$ $(i = 1, 2, 4)$.

In this chapter, we use Beck's method [B].

8.1. In 8.1, let $(\mathcal{E}, \Pi, p)$ be of Diagram 1.6.2 and assume $N \geq 4$. Let W be the Weyl group defined in 2.6 associated to $(\mathcal{E}^\dagger, \Pi^\dagger)$. Let W_0 be a subgroup of W generated by $\{\sigma(i), (1 \leq i \leq n)\}$. Let $\omega_j^\vee \in \oplus_{i=1}^n C\alpha_i^\dagger$ $(1 \leq j \leq n)$ be such that $\frac{2((\alpha_i^\dagger, \omega_j^\vee))}{((\alpha_i^\dagger, \alpha_i^\dagger))} = \delta_{ij}$ $(1 \leq i \leq n)$. Put $P^\vee = \oplus Z\omega_i^\vee$. Define $\bar{W} = W_0 \ltimes P^\vee$ by $(s, x)(s', x') = (ss', s'^{-1}(x) + x')$. We know that there is a certain subgroup $\mathcal{T}$ of Dynkin diagram automorphism of $(\mathcal{E}^\dagger, \Pi^\dagger)$ such that $\bar{W} \cong \mathcal{T} \ltimes W_0$ $(\tau\sigma(i)\tau^{-1} = \sigma(\tau(i))$ $(\tau \in \mathcal{T}))$. If W is of type $A_{N-1}^{(1)}$, then $\mathcal{T} \cong Z/NZ$. For the datum $(\mathcal{E} = (\oplus_{i=1}^N C\bar{\varepsilon}_i) \oplus C\delta \oplus C\lambda_0, \Pi = \{\alpha_i\}, p)$ and $\tau \in \mathcal{T}$, define the datum $(\mathcal{E}^\tau = (\oplus_{i=1}^N C\bar{\varepsilon}_i^\tau) \oplus C\delta \oplus C\lambda_0, \Pi^\tau = \{\alpha_i^\tau\}, p^\tau)$ by

(i) The Dynkin diagram of $(\mathcal{E}^\tau, \Pi^\tau, p^\tau)$ is the same type as the one of $(\mathcal{E}, \Pi, p)$.
(ii) $p^\tau(\alpha_i) = p(\alpha_{\tau^{-1}(i)})$.
(iii) $(\bar{\varepsilon}^\tau_i, \bar{\varepsilon}^\tau_j) = (\bar{\varepsilon}_{\tau^{-1}(i)}, \bar{\varepsilon}_{\tau^{-1}(j)})$ (Here we consider $\tau^{-1}(i)$ under mod$N$).

For $w \in \bar{W}$ and an reduced expression $w = \tau\sigma(i_1)\cdots\sigma(i_r)$, put $(\mathcal{E}^w, \Pi^w, p^w)$ $= (((\,(\mathcal{E}^{\sigma(i_r)})\cdots)^{\sigma(i_1)})^\tau, ((\,(\Pi^{\sigma(i_r)})\cdots)^{\sigma(i_1)})^\tau, ((\,(p^{\sigma(i_r)})\cdots)^{\sigma(i_1)})^\tau)$. Clearly $(\mathcal{E}^w, \Pi^w, p^w)$ doesn't depend on reduced expressions.

Let $U_h(\mathcal{G})'$ be the subalgebra of $U_h(\mathcal{G})$ generated by $\{H_{\alpha_i}, E_i, F_i \ (0 \leq i \leq n)\}$. By Proposition 7.5.1, Lemma 7.5.1 and direct calculation, we have:

Lemma 8.1.1. *Assume that $(\mathcal{E}, \Pi, p)$ is type $A^{(1)}_{N-1}$. Keep notations in 2.3-5. Let $i \in \{0, 1, ..., N-1\}(= Z/NZ)$. Put $K_i = K_{\alpha_i}$.*
(i) There are isomorphisms $T_i : U_h(\mathcal{G}(\mathcal{E}, \Pi, p))' \to U_h(\mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)}))'$ such that (We put $p' = p^{\sigma(i)}$.)

$$\begin{aligned}
&T_iE_i = -\bar{d}'_{i+1}F_iK_i,\ T_iF_i = -\bar{d}'_iK_i^{-1}E_i,\\
&T_iE_{i-1} = q^{-\bar{d}'_i}\bar{d}'_i[\![E_{i-1}, E_i]\!],\ T_iE_{i+1} = q^{-\bar{d}'_{i+1}}(-1)^{p'(\alpha_i)p'(\alpha_{i+1})}\bar{d}'_{i+1}[\![E_{i+1}, E_i]\!],\\
&T_iF_{i-1} = -(-1)^{p'(\alpha_i)p'(\alpha_{i-1})}[\![F_{i-1}, F_i]\!],\ T_iF_{i+1} = -[\![F_{i+1}, F_i]\!].
\end{aligned}$$

$T_i^{-1} : U_h(\mathcal{G}(\mathcal{E}, \Pi, p))' \to U_h(\mathcal{G}(\mathcal{E}^{\sigma(i)}, \Pi^{\sigma(i)}, p^{\sigma(i)}))'$ is given by:

$$\begin{aligned}
&T_i^{-1}E_i = -\bar{d}'_{i+1}K_i^{-1}F_i,\ T_i^{-1}F_i = -\bar{d}'_iE_iK_i,\\
&T_i^{-1}E_{i-1} = q^{-\bar{d}'_i}(-1)^{p'(\alpha_i)p'(\alpha_{i-1})}\bar{d}'_i[\![E_i, E_{i-1}]\!],\ T_i^{-1}E_{i+1} = q^{-\bar{d}'_{i+1}}\bar{d}'_{i+1}[\![E_i, E_{i+1}]\!],\\
&T_i^{-1}F_{i-1} = -[\![F_i, F_{i-1}]\!],\ T_i^{-1}F_{i+1} = -(-1)^{p'(\alpha_i)p'(\alpha_{i+1})}[\![F_i, F_{i+1}]\!].
\end{aligned}$$

For $\tau \in \mathcal{T}$, there is an isomorphism $T_\tau : U_h(\mathcal{G}(\mathcal{E}, \Pi, p))' \to U_h(\mathcal{G}(\mathcal{E}^\tau, \Pi^\tau, p^\tau))'$ such that $T_\tau(H_{\alpha_i}) = H_{\alpha_{\tau(i)}}$, $T_\tau(E_i) = E_{\tau(i)}$, $T_\tau(F_i) = F_{\tau(i)}$.

(ii) T_i's satisfy Braid relation:

$$T_iT_j = T_jT_i\,((\alpha_i, \alpha_j) = 0), \quad T_iT_jT_i = T_jT_iT_j\,(|(\alpha_i, \alpha_j)| = 1).$$

It also hold that $T_\tau T_i T_\tau^{-1} = T_{\tau(i)}$.

(iii) By (ii), putting $T_w = T_\tau T_{i_1}\cdots T_{i_r}$ for $w \in \bar{W}$ whose reduced expression is $w = \tau\sigma(i_1)\cdots\sigma(i_r)$, T_w is well-defined. Moreover we have:

$$T_w(E_i) = E_j,\ T_w(F_i) = F_j \quad \textit{if } w(\alpha_i) = \alpha_j.$$

There is an C-anti-automorphism Ω such that

$$\Omega(E_i) = \bar{d}_{i+1}F_i,\ \Omega(E_i) = \bar{d}_{i+1}F_i,\ \Omega(H) = H,\ \Omega(h) = -h\,.$$

Moreover $\Omega T_w = T_w\Omega$ $(w \in \bar{W})$.

8.2. Put $T_{\omega_i} = T_{\omega_i^\vee}$. For $1 \le i \le n$, $k > 0$ and $s \in Z$, let

$$\bar{\psi}_{ik}^{(s)} = K_\delta^{-\frac{k}{2}} q^{-(\alpha_i,\alpha_i)} [\![T_{\omega_i}^s(E_i), T_{\omega_i}^{k+s}(K_i^{-1}F_i)]\!] . \qquad (8.1.1)$$

Put $Q_{ij,k} = \frac{q^{k(\alpha_i,\alpha_j)} - q^{-k(\alpha_i,\alpha_j)}}{q-q^{-1}}$ and $\dot{C}_{ij} = q^{(\alpha_i,\alpha_j)} K_\delta^{-\frac{1}{2}}$. By [B], we have:

Lemma 8.2.1. *(i)* $K_\delta^{\frac{1}{k}} \bar{\psi}_{ik}^{(s)} \in \mathcal{N}_+$ *if* $s \le 0$ *and* $k+s > 0$.
(ii) Assume $p(\alpha_i) = 0$. *Let* $r > 0$ *and* $m \in Z$. *Then* $\bar{\psi}_{ir}^{(s)} = \bar{\psi}_{ir}^{(s')}$ $(s, s' \in Z)$ *and*

$$[\bar{\psi}_{ir}^{(s)}, T_{\omega_i}^m(F_i)] = -K_\delta^{\frac{1}{2}} Q_{ii,1} \Big\{ ((q-q^{-1}) \sum_{k=1}^{r-1} \dot{C}_{ii}^{1-k} T_{\omega_i}^{m+k}(F_i) \bar{\psi}_{i,r-k}^{(s)}) + \dot{C}_{ii}^{1-r} T_{\omega_i}^{m+r}(F_i) \Big\},$$

$$[\bar{\psi}_{ir}^{(s)}, T_{\omega_i}^m(E_i)] = K_\delta^{-\frac{1}{2}} Q_{ii,1} \Big\{ ((q-q^{-1}) \sum_{k=1}^{r-1} \dot{C}_{ii}^{k-1} T_{\omega_i}^{m-k}(E_i) \bar{\psi}_{i,r-k}^{(s)}) + \dot{C}_{ii}^{r-1} T_{\omega_i}^{m-r}(E_i) \Big\}.$$

(ii) Assume $1 \le i \ne j \le n$. *Let* $r > 0$ *and* $m \in Z$. *Then:*

$$\begin{aligned} &[\bar{\psi}_{ir}^{(s)}, T_{\omega_j}^m(F_j)] \\ &\quad = K_\delta^{\frac{1}{2}} Q_{ij,1} \Big\{ ((q-q^{-1}) \sum_{k=1}^{r-1} (-\dot{C}_{ij})^{1-k} T_{\omega_j}^{m+k}(F_j) \bar{\psi}_{i,r-k}^{(s)}) + (-\dot{C}_{ij})^{1-r} T_{\omega_j}^{m+r}(F_j) \Big\}, \\ &[\bar{\psi}_{ir}^{(s)}, T_{\omega_j}^m(E_j)] \\ &\quad = -K_\delta^{-\frac{1}{2}} Q_{ij,1} \Big\{ ((q-q^{-1}) \sum_{k=1}^{r-1} (-\dot{C}_{ij})^{k-1} T_{\omega_j}^{m-k}(E_j) \bar{\psi}_{i,r-k}^{(s)}) + (-\dot{C}_{ij})^{r-1} T_{\omega_j}^{m-r}(E_j) \Big\}. \end{aligned}$$

Let $o(i) \in \{\pm 1\}$ satisfy that $o(i) \ne o(j)$ if $(\alpha_i, \alpha_j) \ne 0$ $(i \ne j)$. Put $\hat{T}_{\omega_i}^m E_i = o(i)^m T_{\omega_i}^m E_i$ and $\hat{T}_{\omega_i}^m F_i = o(i)^m T_{\omega_i}^m F_i$. Define $\hat{\bar{\psi}}_{ir}^{(s)}$ by replacing T_{ω_i} of (8.1.1) with $\hat{T}_{\omega_i}$. By Lemma 8.2.1, we have:

Lemma 8.2.2. *Assume* $j \ne i$ *or* $p(\alpha_i) = 0$. *Then:*

$$\begin{aligned} &[\hat{\bar{\psi}}_{ir}^{(s)}, \hat{T}_{\omega_j}^m(F_j)] \\ &\quad = -K_\delta^{\frac{1}{2}} Q_{ij,1} \Big\{ ((q-q^{-1}) \sum_{k=1}^{r-1} \dot{C}_{ij}^{1-k} \hat{T}_{\omega_j}^{m+k}(F_j) \hat{\bar{\psi}}_{i,r-k}^{(s)}) + \dot{C}_{ij}^{1-r} \hat{T}_{\omega_j}^{m+r}(F_j) \Big\}, \\ &[\hat{\bar{\psi}}_{ir}^{(s)}, \hat{T}_{\omega_j}^m(E_j)] \\ &\quad = K_\delta^{-\frac{1}{2}} Q_{ij,1} \Big\{ ((q-q^{-1}) \sum_{k=1}^{r-1} \dot{C}_{ij}^{k-1} \hat{T}_{\omega_j}^{m-k}(E_j) \hat{\bar{\psi}}_{i,r-k}^{(s)}) + \dot{C}_{ij}^{r-1} \hat{T}_{\omega_j}^{m-r}(E_j) \Big\}. \end{aligned}$$

Define $h_{ik}^{(s)} \in U_h(\mathcal{G})$ $(k > 0)$ by the following generating function in z.

$$\exp((q-q^{-1}) \sum_{k=1}^{\infty} h_{ik}^{(s)} z^k) = 1 + (q-q^{-1}) \sum_{k=1}^{\infty} \hat{\bar{\psi}}_{ik}^{(s)} z^k.$$

Remark. For $(\alpha_i, \alpha_i) = 0$, we have not shown $[\hat{\bar{\psi}}_{ik}^{(s)}, \hat{\bar{\psi}}_{ir}^{(s)}] = 0$ yet. Hence an uncertainly of the definition of $h_{ik}^{(s)}$ has still remined. It depends on an order of $\{\hat{\bar{\psi}}_{ik}^{(s)}\}$.

By Lemma 8.2.2, we have:

Lemma 8.2.3. *Assume $j \neq i$ or $(\alpha_i, \alpha_i) \neq 0$. Then:*

$$\begin{aligned} [h_{ik}^{(s)}, \hat{T}_{\omega_j}^m(F_j)] &= -\tfrac{1}{k} Q_{ij,k} K_\delta^{\frac{k}{2}} \hat{T}_{\omega_j}^{m+k}(F_j), \\ [h_{ik}^{(s)}, \hat{T}_{\omega_j}^m(E_j)] &= \tfrac{1}{k} Q_{ij,k} K_\delta^{-\frac{k}{2}} \hat{T}_{\omega_j}^{m-k}(E_j). \end{aligned}$$

8.3. Lemma 8.3.1. *Let $1 \leq i \leq n$ and $r \in Z$. Then:*

$$\begin{aligned} &[\![T_{\omega_i}^{m+r}(F_i), T_{\omega_i}^m(F_i)]\!] = -[\![T_{\omega_i}^{m+1}(F_i), T_{\omega_i}^{m+r-1}(F_i)]\!], \\ &[\![T_{\omega_i}^{m}(E_i), T_{\omega_i}^{m+r}(E_i)]\!] = -[\![T_{\omega_i}^{m+r-1}(E_i), T_{\omega_i}^{m+1}(E_i)]\!]. \end{aligned}$$

Proof. For $(\alpha_i, \alpha_i) \neq 0$, we have already known these by [B]. For $(\alpha_i, \alpha_i) = 0$, by Lemma 8.2.3 and $T_{\omega_i}^m(F_i)^2 = T_{\omega_i}^m(E_i)^2 = 0$,

$$[T_{\omega_i}^{m+r}(F_i), T_{\omega_i}^m(F_i)] = [T_{\omega_i}^{m+r}(E_i), T_{\omega_i}^m(E_i)] = 0, \tag{8.3.1}$$

which are nothing else but the formulae we want.

Q.E.D.

Lemma 8.3.2. *Let $(\alpha_i, \alpha_i) = 0$ and $r > 0$. Then $\bar{\psi}_{ir}^{(s)} = \bar{\psi}_{ir}^{(s')}$ and*

$$[h_{ir}^{(s)}, T_{\omega_i}^m(F_i)] = [h_{ir}^{(s)}, T_{\omega_i}^m(E_i)] = 0. \tag{8.3.2}$$

Proof. By (8.3.1), we have:

$$[E_i, \bar{\psi}_{ir}^{(0)}] = [\bar{\psi}_{ir}^{(0)}, F_i] = 0. \tag{8.3.3}$$

We use an induction on r. Let $1 \leq j \leq n$ be such that $(\alpha_i, \alpha_j) \neq 0$. First we assume $r = 1$. Then $h_{i1}^{(s)} = o(i)\bar{\psi}_{i1}^{(s)}$.

$$\begin{aligned} \bar{\psi}_{i1}^{(-1)} &= K_\delta^{-\frac{1}{2}} [T_{\omega_i}^{-1}(E_i), K_i^{-1} F_i] \\ &= o(i) K_\delta^{-\frac{1}{2}} Q_{ji,1}^{-1} K_\delta^{\frac{1}{2}} \; [[h_{j1}^{(0)}, E_i], K_i^{-1} F_i] \quad \text{(by Lemma 8.2.3)} \\ &= o(i) Q_{ji,1}^{-1} K_i^{-1} \cdot o(i) Q_{ji,1} [K_\delta^{-\frac{1}{2}} T_{\omega_i}(F_i), E_i] \\ &= K_\delta^{-\frac{1}{2}} [T_{\omega_i}(K_i^{-1} F_i), E_i] \\ &= \bar{\psi}_{i1}^{(0)}. \end{aligned}$$

Hence, by (8.3.3), we get our formulae for $r = 1$.

We assume that we have shown the lemma for 1, 2,..., $r-1$. Firstly we show $[h_{j1}^{(0)}, h_{jr-1}^{(0)}] = 0$. By Lemma 8.2.3, we have:

$$[[h_{j1}^{(0)}, h_{jr-1}^{(0)}], \hat{T}_{\omega_k}^m(F_k)] = [[h_{j1}^{(0)}, h_{jr-1}^{(0)}], \hat{T}_{\omega_k}^m(E_k)] = 0$$

for $1 \le k \le n$ and $m \in Z$. By Lemma 8.2.1 (i), $K_\delta^{\frac{r}{2}}[h_{j1}^{(0)}, h_{jr-1}^{(0)}] \in N^+$. We know the fact that $\hat{T}_{\omega_k}^m(F_k)$, $\hat{T}_{\omega_k}^m(E_k)$ and $\mathcal{H}$ generate $U_h(\mathcal{G})$. Hence, by Proposition 6.2.1, we get

$$[h_{j1}^{(0)}, h_{jr-1}^{(0)}] = 0 \tag{8.3.4}$$

as well as $[h_{j1}^{(0)}, \bar{\psi}_{jr-1}^{(0)}] = 0$. Hence:

$$\begin{aligned}
\bar{\psi}_{ir}^{(-1)} &= K_\delta^{-\frac{1}{2}}[T_{\omega_i}^{-1}(E_i), T_{\omega_i}^{r-1}(K_i^{-1}F_i)] \\
&= o(i)K_\delta^{-\frac{r}{2}}Q_{ji,1}^{-1}K_\delta^{\frac{1}{2}}[[h_{j1}^{(0)}, E_i], T_{\omega_i}^{r-1}(K_i^{-1}F_i)] \quad \text{(by Lemma 8.2.3)} \\
&= o(i)Q_{ji,1}^{-1}K_\delta^{\frac{1-r}{2}}\Big\{[h_{j1}^{(0)}, K_\delta^{\frac{r-1}{2}}\bar{\psi}_{ir-1}^{(0)}] + o(i)Q_{ji,1}[K_\delta^{-\frac{1}{2}}T_{\omega_i}^r(K_i^{-1}F_i), E_i]\Big\} \\
&= \bar{\psi}_{ir}^{(0)}.
\end{aligned}$$

Hence, by (8.3.3), we get our formulae.

Q.E.D.

We put $h_{ir} = h_{ir}^{(s)}$ and $\hat{\bar{\psi}}_{ik}\hat{\bar{\psi}}_{ik}^{(s)}$ $(r > 0, 1 \le i \le n)$ which is well defined by Lemma 8.3.2. Similarly to show (8.3.4), we have:

Lemma 8.3.3. $[h_{ir}, h_{ir'}] = 0$.

By Lemma 8.2.3 and Lemma 8.3.2, we have:

Lemma 8.3.4.

$$\begin{aligned}
[h_{ik}, \hat{T}_{\omega_j}^m(F_j)] &= -\tfrac{1}{k}Q_{ij,k}K_\delta^{\frac{k}{2}}\hat{T}_{\omega_j}^{m+k}(F_j), \\
[h_{ik}, \hat{T}_{\omega_j}^m(E_j)] &= \tfrac{1}{k}Q_{ij,k}K_\delta^{-\frac{k}{2}}\hat{T}_{\omega_j}^{m-k}(E_j).
\end{aligned}$$

We know that $K_\delta^{\frac{k}{2}}h_{ik} \in N^+$. By Lemma 8.3.4 and Proposition 6.2.1, since $\hat{T}_{\omega_j}^m(E_j)$, $\hat{T}_{\omega_j}^m(F_j)$ and $\mathcal{H}$ generate $U_h(\mathcal{G})$, we have:

Lemma 8.3.5. *Keep notations in 1.6. If* $\sum_{i=1}^N \bar{d}_i = 0$, *then*

$$\sum_{i=1}^n \Big[\sum_{j=1}^i k\bar{d}_j\Big] K_\delta^{\frac{k}{2}} h_{ik} = 0 \quad (k \ge 1). \tag{8.3.5}$$

in $\mathcal{N}^+ \subset U_h(\mathcal{G})(\mathcal{E},\Pi,p)$ *of* $(\mathcal{E},\Pi,p)$ *of Diagram 1.6.2* $(N \geq 4)$.

After all we have:

Theorem 8.3.6. *Let* $(\mathcal{E},\Pi,p)$ *be the datum of Diagram 1.6.2* $(N \geq 4)$ *with* $\sum \bar{d}_i = 0$. *Then the defining relations of* $U_h(\mathcal{G}(\mathcal{E},\Pi,p))$ *are defined by adding (8.3.5) to the ones of Theorem 6.6.1.*

8.4. For $1 \leq i \leq N-1$ and $r \geq 0$, put

$$\psi_{ir} = \begin{cases} (q-q^{-1})K_i\hat{\bar{\psi}}_{ik} & (r>0), \\ K_i & (r=0). \end{cases}$$

and $\varphi_{ir} = \Omega(\psi_{ir})$. Put $h_{i,-r} = h_{ir}$ $(r>0)$. For $1 \leq i \leq N-1$ and $k \in Z$, put $x^-_{ik} = \hat{T}^k_{\omega_i}(F_i)$ and $x^+_{ik} = \hat{T}^{-k}_{\omega_i}(E_i)$. Similar to [B], we have:

Theorem 8.4.1. *Let* $(\mathcal{E},\Pi,p)$ *be the datum of Diagram 1.6.2* $(N \geq 3)$. *Then* $U_h(\mathcal{G}(\mathcal{E},\Pi,p))$ *is defined with the generators* $\{H \in \mathcal{H},\ x^\pm_{ij},\ h_{ik}\}$ *and the relations:*

$$[H, x^\pm_{jk}] = (\pm\alpha_j + k\delta)(H)x^\pm_{jk},$$
$$[h_{ik}, h_{jl}] = \delta_{k,-l}\tfrac{1}{k}Q_{ij,k}\tfrac{K^k_\delta - K^{-k}_\delta}{q-q^{-1}},$$
$$x^\pm_{ik+1}x^\pm_{jl} - (-1)^{p(\alpha_i)p(\alpha_j)}q^{\pm(\alpha_i,\alpha_j)}x^\pm_{jl}x^\pm_{ik+1} = (-1)^{p(\alpha_i)p(\alpha_j)}q^{\pm(\alpha_i,\alpha_j)}x^\pm_{ik}x^\pm_{jl+1} - x^\pm_{jl+1}x^\pm_{ik},$$
$$[x^+_{ik}, x^-_{jl}] = \delta_{ij}\tfrac{K_\delta^{\frac{k-l}{2}}\psi_{ik+l} - K_\delta^{\frac{l-k}{2}}\phi_{ik+l}}{q-q^{-1}},$$

$[x^\pm_{ik}, x^\pm_{il}] = 0$ $\qquad$ *if* $(\alpha_i,\alpha_i) = 0$,

(In the following equations, $Sym_{k_1,k_2,\ldots,k_s}$ *means symmetrization with respect to* $\{k_1,k_2,\ldots,k_s\}$.*)*

$Sym_{k_1,k_2}[\![x^\pm_{ik_1},[\![x^\pm_{ik_2},x^\pm_{jl}]\!]]\!] = 0$ $\qquad$ *if* $(\alpha_i,\alpha_i) \neq 0$ *and* $(\alpha_i,\alpha_j) \neq 0$,

$Sym_{k_1,k_2}[[\![[\![x^\pm_{il},x^\pm_{jk_1}]\!],x^\pm_{um}]\!],x^\pm_{jk_2}] = 0$ $\qquad$ *if* $\overset{i}{\times}$ — $\overset{j}{\otimes}$ — $\overset{u}{\times}$

(Each of the following equations means an equation as a generate function in an indeterminate z.)

$$\textstyle\sum_{k\geq 0}\psi_{ik}z^k = K_i\exp((q-q^{-1})\sum_{r\geq 1}h_{ir}z^r),$$
$$\textstyle\sum_{k\geq 0}\phi_{ik}z^k = K_i^{-1}\exp((q^{-1}-q)\sum_{r\geq 1}h_{i,-r}z^r).$$

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